THE TWELFTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Twelfth Annual Meeting of the Society was held in New York City on Thursday and Friday, December 28–29, 1905. The interest of the occasion was greatly increased by the simultaneous meetings of the American Physical Society and the Astronomical and Astrophysical Societies of America. The large attendance and the extensive programmes gave additional importance to an otherwise noteworthy scientific convention. Community of interest received due attention. A joint session of the Mathematical and Physical Societies was held on Friday afternoon for the purpose of hearing Professor V. F. Bjerknes, of the University of Stockholm, who spoke on "Experimental demonstration of hydrodynamic action at a distance." On Friday evening nearly ninety representatives of the three societies attended a dinner organized in honor of Professor Bjerknes. The common luncheon between each day's sessions afforded an excellent opportunity to renew and make acquaintance and to compare notes scientific or otherwise. Informal gatherings were also held on Thursday evening.

The attendance at the four sessions of the Mathematical Society, which exceeded that of any previous meeting, included the following sixty-six members:

Professor Cleveland Abbe, Professor O. P. Akers, Miss Grace Andrews, Professor G. A. Bliss, Professor E. W. Brown, Dr. W. H. Bussey, Dr. J. E. Clark, Miss Emily Coddington, Professor F. N. Cole, Miss E. B. Cowley, Professor E. S. Crawley, Professor D. R. Curtiss, Dr. W. S. Dennett, President E. A. Engler, Professor T. C. Esty, Professor H. B. Fine, Professor B. F. Finkel, Mr. A. B. Frizell, Professor A. S. Gale, Miss A. B. Gould, Miss Ida Griffiths, Professor F. H. Hodge, Professor E. V. Huntington, Professor J. I. Hutchinson, Mr. S. A. Joffe, Dr. Edward Kasner, Professor O. D. Kellogg, Professor C. J. Keyser, Professor Gustave Legras, Professor E. O. Lovett, Professor James Maclay, Professor H. P. Manning, Professor Max Mason, Professor Helen A. Merrill, Professor Mansfield Merriman, Dr. C. L. E. Moore, Professor G. D. Olds, Professor W. F. Osgood, Dr. J. L. Patterson, Professor Alexander Pell, Professor A. W. Phillips,

Professor James Pierpont, Mr. R. G. D. Richardson, Dr. F. H. Safford, Professor Arthur Schultze, Professor Charlotte A. Scott, Dr. C. H. Sisam, Dr. Clara E. Smith, Professor P. F. Smith, Dr. H. F. Stecker, Professor W. E. Story, Dr. C. E. Stromquist, Professor H. D. Thompson, Miss M. E. Trueblood, Professor H. W. Tyler, Miss A. L. Van Benschoten, Professor J. M. Van Vleck, Professor Oswald Veblen, Professor L. A. Wait, Mr. H. E. Webb, Professor J. B. Webb, Professor A. G. Webster, Mr. W. D. A. Westfall, Professor H. S. White, Dr. E. B. Wilson, Mr. J. E. Wright.

The President of the Society, Professor W. F. Osgood, occupied the chair, being relieved at the Friday afternoon session by Professor E. W. Brown. President Carl Barus, of the Physical Society, presided at the joint session. The Council announced the election of the following persons to membership in the Society: Mr. R. L. Börger, University of Missouri; Professor W. B. Carver, Ursinus College; Mr. A. J. Champreux, University of California; Dr. Emily Coddington, New York, N. Y.; Dr. F. J. Dohmen, University of Texas; Professor O. E. Glenn, Drury College; Mr. E. S. Haynes, University of Missouri; Professor J. H. Jeans, Princeton University; Mr. A. R. Maxson, Columbia University; Professor J. F. Travis, Georgia School of Technology; Professor Vito Volterra, University of Rome; Miss May E. G. Waddell, Orono, Canada. Nineteen applications for admission to membership in the Society were received.

Reports were received from the Treasurer, Librarian, and Auditing Committee. These reports will appear in the Annual Register, now in press. The membership of the Society has increased during the past year from 473 to 512. The number of papers presented at all meetings during the year was 147, as against 148 in 1904. The total attendance of members was 280; 161 members attended at least one meeting during the year. The library now contains about 2,000 bound volumes; a list of the journals and of the accessions during 1905 is printed in the Annual Register. The Treasurer's report shows a balance of $3,833.01 on hand December 16, 1905; of this balance $2,132.58 is credited to the life-membership fund.

At the annual election, which closed on Friday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor CHARLOTTE A. SCOTT,
	Professor IRVING STRINGHAM,
Secretary,	Professor F. N. COLE,
Treasurer,	Dr. W. S. DENNETT,
Librarian,	Professor D. E. SMITH,

Committee of Publication,

Professor F. N. COLE,
Professor ALEXANDER ZIWET,
Professor D. E. SMITH.

Members of the Council to serve until December, 1908,

Professor C. L. BOUTON,	Dr. EDWARD KASNER,
Professor L. E. DICKSON,	Professor E. J. TOWNSEND.

The following papers were read at the meeting:

(1) Mr. R. G. D. RICHARDSON: "Multiple improper integrals."

(2) Mr. A. B. FRIZELL: "On the continuum problem" (preliminary communication).

(3) Professor D. R. CURTISS: "The vanishing of the wronskian and the problem of linear dependence."

(4) Professor J. I. HUTCHINSON: "On certain automorphic groups whose coefficients are integers in a quadratic field."

(5) Professor E. V. HUNTINGTON: "Note on the fundamental propositions of algebra" (preliminary communication).

(6) Professor C. J. KEYSER: "Concerning a self-reciprocal plane geometry."

(7) Dr. C. L. E. MOORE: "Geometry of circles orthogonal to a given sphere."

(8) Dr. EDWARD KASNER: "Invariants of differential elements for arbitrary point transformation."

(9) Mr. A. B. FRIZELL: "A method of building up the fundamental operation groups of arithmetic."

(10) Professor G. A. BLISS: "A proof of the fundamental theorem of analysis situs."

(11) Professor O. P. AKERS: "On the congruence of axes in a bundle of linear line complexes."

(12) Dr. PETER FIELD: "Note on certain groups of transformations of the plane into itself."

(13) Mr. GEORGE PEIRCE: "A new approximate construction for π."

(14) Professor MAX MASON: "Curves of minimum moment of inertia."

(15) Professor A. G. WEBSTER: "Application of a definite integral involving Bessel's functions to the self-inductance of solenoids."

(16) Mr. J. E. WRIGHT: "Correspondences and the theory of continuous groups."

(17) Mr. J. E. WRIGHT: "An application of the differential invariants of space."

(18) Dr. CLARA E. SMITH: "A theorem of Abel and its application to the development of an arbitrary function in terms of Bessel's functions."

(19) Professor V. F. BJERKNES: "Experimental demonstration of hydrodynamic action at a distance."

(20) Dr. R. P. STEPHENS: "On the pentadeltoid."

(21) Professor M. I. PUPIN: "Establishment of the steady state in a sectional wave conductor."

(22) Dr. J. J. QUINN: "A linkage for the kinematic description of a cissoid."

Dr. Stephens was introduced by Professor Morley. Mr. Peirce's paper was communicated to the Society through Professor E. W. Brown, Dr. Quinn's paper through Dr. A. L. Baker. In the absence of the authors Mr. Peirce's paper was read by Professor Brown, and the papers of Dr. Field, Professor Pupin and Dr. Quinn were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The method used by Mr. Richardson for defining multiple improper integrals differs from that of Professor Pierpont (*Transactions*, January, 1906). Analogous to the definition given by Vallée Poussin (*Journal de Mathématiques*, series 4, volume 8, 1892) for one variable, a truncated limited function $f_{\lambda_1\lambda_2}$ of m variables $x_1, \cdots, x_m$ is defined over a limited aggregate $\mathfrak{A}$. The integral $I_{\lambda_1\lambda_2}$ of this truncated function always exists, and if $\lim_{\lambda_1=\infty\,\lambda_2=\infty} I_{\lambda_1\lambda_2}$ exists, it is called the improper integral of f in $\mathfrak{A}$. The integrals defined by this method are absolutely convergent. Mr. Richardson has shown that the integrals defined by Professor Pierpont are also absolutely convergent and that the two integrals are identical. Various properties, theorems of the

mean, and convergent tests are derived from this general integral. If the variables are broken up into two sets $x_1, \cdots, x_n$ and $x_{n+1}, \cdots, x_m$, and if $\mathfrak{B}$ and $\mathfrak{C}$ are the n-way and $(m-n)$-way aggregates corresponding to $\mathfrak{A}$, we may call

$$J(x_1, \cdots, x_n) = \int_{\mathfrak{C}} f(x_1, \cdots, x_m)$$

an integral depending on the parameters $x_1, \cdots, x_n$, and investigate it with regard to continuity, differentiation, and integration. If J is an integral uniformly convergent in $\mathfrak{B}$, it is a continuous function of $x_1, \cdots, x_n$. If the integral of f over $\mathfrak{C}$ is regularly convergent, and the integral of f over $\mathfrak{B}$ is uniformly convergent we may write

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f(x_1, \cdots, x_m) = \int_{\mathfrak{C}} \int_{\mathfrak{B}} f(x_1, \cdots, x_m).$$

Various other criteria for differentiating under the integral sign and for reduction of multiple integrals are developed.

2. Mathematicians have naturally looked to the problems of counting the points of the linear continuum or of arranging them in a well ordered set for applications of Cantor's transfinite numbers, and perhaps the most general opinion is that the number of points on a segment of a straight line will turn out to be either Aleph-eins or some subsequent Aleph. Mr. Frizell's paper adopts $\aleph_0^{\aleph_0}$ for the cardinal number of the continuum and considers the possibility of this expression being equal to Aleph-null. This leads to a scrutiny of the definition of involution for transfinite numbers, in view of the remarkable relations

$$\aleph_0 = \aleph_0^2 = \aleph_0^3 = \cdots = \aleph_0^\nu,$$

$$2^{\aleph_0} = 3^{\aleph_0} = \cdots = \mu^{\aleph_0} = \cdots = \aleph_0^{\aleph_0}.$$

The writer's opinion, however, is that no satisfactory solution is likely to be reached except by exhibiting the continuum as a well ordered class possessing a definite rank in Cantor's series of ordinal types. A method of doing this is suggested by first well ordering the class of all finite simple continued fractions and then ordering all possible infinite sequences composed of

members of this class according to the place in the scheme of the elements omitted in their formation. The writer hopes to develop this process more fully in a subsequent communication.

3. Although the vanishing of the wronskian is both a necessary and sufficient condition for the linear dependence of n analytic functions $u_1, u_2, \cdots, u_n$, this is by no means always the case for sets of functions of a real variable which are subject to no other restriction than that they have finite derivatives of order $> n-2$ in the interval under consideration. In Professor Curtiss's paper a theorem concerning the vanishing of the wronskian in an infinite set of points is established, and by its aid new criteria for linear dependence are obtained.

4. The groups considered by Professor Hutchinson are formed by the linear unimodular transformations of a single variable, the coefficients being of the form $\alpha + \lambda\alpha'$ in which λ is a root of the equation $\lambda^2 - m\lambda + n = 0$, and α, α', m, n are integers. Linear relations among the coefficients are assumed so as to reduce the number of arbitrary integers $\alpha, \alpha', \cdots$ involved in the coefficients to four, among which a quadratic relation exists on account of the determinant of the transformation being unity. The main object of the paper is to determine the forms of these relations so that the transformations shall form a group properly discontinuous in the plane of the transformed variable.

5. In Professor Huntington's paper the attempt is made to present sets of postulates for the various branches of real algebra in a form available for pedagogical purposes. For example, the fundamental propositions selected for the algebra of all real numbers are the following, the operations of addition and multiplication and the relation of order being taken as the fundamental concepts:

*A*1. Every two elements a and b determine uniquely an element $a + b$, called their sum.

*A*2. $(a + b) + c = a + (b + c)$.

*A*3. $a + b = b + a$.

*A*4. If $a + x = a + y$, then $x = y$.

*A*5. There is an element z such that $z + z = z$. This element z proves to be uniquely determined, and is called 0.

*A*6. For every element a there is an element a' such that $a + a' = 0$.

$M1$. Every two elements a and b determine uniquely an element ab, called their product; and if $a \neq 0$ and $b \neq 0$, then $ab \neq 0$.*

$M2$. $(ab)c = a(bc)$.

$M3$. $ab = ba$.

$M4$. If $ax = ay$, and $a \neq 0$, then $x = y$.

$M5$. There is an element u, different from 0, such that $uu = u$. This element u proves to be uniquely determined, and is called 1.

$M6$. For every element a, not 0, there is an element a'' such that $aa'' = 1$.

$AM1$. $a(b + c) = ab + ac$.

$S1$. If $a \neq b$, then either $a < b$ or $b < a$.

$S2$. If $a < b$, then $a \neq b$.

$S3$. If $a < b$ and $b < c$, then $a < c$.

$S4$. Dedekind's postulate.

$SA1$. If $x < y$ then $a + x < a + y$.

$SM1$. If $a > 0$ and $b > 0$, then $ab > 0$.

The sets of postulates for the algebra of positive reals (with or without 0), the algebra of all integers, the algebra of positive integers (with or without 0), etc., are obtained by replacing one or more of the postulates $A5$, $A6$, $M5$, $M6$ by their opposites, and slightly modifying $SA1$ and $SM1$.

6. Of the infinitely many possible geometries of a given analytic type, all but a few deal with configurations too remote from (present) intuition to engage the interest and thereby to justify or invite detailed investigation and exposition. Accordingly it is a matter of judgment to select from possible geometries of a type those that are worthy of such attention. That determination once effected, or even in course of it, interest attaches chiefly to the general comparative anatomy of the allied theories rather than to a minute or histological study of a single one of the variety. Professor Keyser's paper adds another theory to the list of the noteworthy doctrines of the type of the Plücker line geometry. It is a plane geometry in which the circle range (one parameter circle system determined by two circles) is employed as element. The self-reciprocal character of the geometry is evident in the fact that the element is equally determinable as the intersection of two circle congruences,

* The latter part of $M1$ is redundant, in view of $A4$ and $AM1$.

where by congruence is meant the system of circles orthogonal to a given circle. The circle and the circle congruence are reciprocal elements in the circle geometry of the plane. The circle range can be regarded either as a linear "locus" of circles or as a linear "envelope" of a pencil of circle congruences. The circles and circle congruences are related as pole and polar with respect to the invariant circle configuration composed of the point circles of the plane.

7. If p, p' are the point spheres of the pencil determined by the fixed sphere S and the variable plane P, as P rotates about a line l, a circle c orthogonal to S will be traced by the points p, p'. By means of this transformation Dr. Moore discusses the geometry of circles orthogonal to a given sphere.

8. The simplest example of an expression which is invariant with respect to arbitrary point transformation is the anharmonic ratio of the tangents to four curves passing through a common point. The invariance here results from the fact that lineal elements with a common point undergo homographic transformation. Dr. Kasner determines all invariants depending upon the curvature as well as the direction of any number of curves passing through a point. (The results obtained do not agree with those given by Rabut in a memoir published in the *Journal de l'Ecole Polytechnique* for 1898). The eight-parameter group induced on a bundle of curvature elements is studied in detail. A representation in space of four dimensions suggests a classification of differential equations of the second order according to *rank* — a classification which is invariant with respect to point transformation. Equations of the first rank have been studied by Lie and R. Liouville, but the others have apparently not been noticed. The group of all conformal transformations is found to induce a four-parameter group; it leads to absolute invariants not always expressible in terms of angles.

9. The problem of establishing the laws of algebra on a logical basis has two aspects. One is their reduction to a working system of postulates; this is discussed in Professor Huntington's paper. The other, which forms the subject of Mr. Frizell's second paper is that of providing a means for readily obtaining the proofs. This is effected chiefly by aid of the following proposition in the abstract theory of groups. Let

C denote a class whose members admit of comparison as equal or not equal, and suppose that C constitutes an abelian semigroup with respect to a certain rule of combination denoted by $\circ$. Form the class $K = \{(m, n)\}$ of pairs of members of C. In K, declare $(m, q) = (n, p)$ if $m \circ p = n \circ q$ and lay down a rule $\odot$ whereby $(m, q) \odot (n, r) = (m \circ n, q \circ r)$. Then it can be shown that K is a C-class, $K = C_1$, and that $\odot$ denotes a rule of combination for C_1 such that C_1 forms a group with respect to $\odot$. By three applications of this theorem we obtain from the class of natural numbers $C = \{n\}$ first the class of absolute rational numbers $C_1 = \{R\}$, then the whole class of absolute numbers $C_2 = \{X\}$, and finally the class of complex numbers $C_3 = \{Z\}$ not including zero. The process consists in representing the successive classes geometrically in the first cases by the circumferences of a set of concentric circles, in the last case by points in a plane. The method admits of extension to the development of the number system of quaternions.

10. Professor Bliss gave a proof of the theorem that a continuous closed curve without double points and with a continuously turning tangent divides the plane into two regions, an interior and an exterior. The proofs heretofore given of the division of the plane by a Jordan curve have been of two kinds, those in which only the continuity of the curve has been assumed, and those in which generality in the assumption has to some extent been sacrificed in order to attain greater simplicity of proof. The present proof is of the second kind.

11. In Professor Akers's paper the lines of the (3, 2) congruence, made up of the axes in the bundle of complexes $\sigma_1(p_{23} - e_1 p_{41}) + \sigma_2(p_{31} - e_2 p_{42}) + \sigma_3(p_{12} - e_3 p_{43}) = 0$, were arranged on ∞^1 ruled surfaces of order five. The σ_i are regarded as point coördinates in the plane at infinity; and in this way the configuration is mapped on the plane. The points σ of the plane at infinity and the lines of the congruence are in (1, 1) correspondence. The condition was obtained under which the complex to which the surfaces belong becomes special, and all the exceptional lines were determined. The six congruences having the same focal surface were discussed and compared.

12. Dr. Field's note deals with certain groups of transformations of the plane into itself, which were suggested in connec-

tion with the study of a plane quintic curve with five cusps. The paper is published in full in the present number of the BULLETIN.

13. Mr. George Pierce's paper contained an approximate construction for π which gives the value of this constant with an error of .000048. It compares very favorably with all previous constructions as to both simplicity and accuracy.

14. Professor Mason discussed the problem of determining a plane curve joining two given points whose moment of inertia with respect to a given point or a given line is a minimum. In the first case the extremals are expressed by trigonometric functions, in the second by elliptic functions. Both problems possess discontinuous solutions for certain positions of the given end points.

15. In Professor Webster's paper the solenoid is treated as a cylindrical current sheet, and the self-inductance found from that of an infinite sheet by determining the correction for the ends. For this purpose use is made of a result of H. Weber, that the potential due to a uniform circular disk of radius α at a point whose coördinates are z, x is

$$U = 2\pi\alpha \int_0^\infty \frac{e^{-\lambda z}}{\lambda} J_0(\lambda x) J_1(\lambda\alpha)\, d\lambda.$$

From this the axial field is obtained by differentiating with respect to z, the flux by integrating with respect to x, introducing $J_1(\lambda\alpha)$ a second time as a factor. The integration with respect to λ and z may then be carried out after changing the order of integration, giving a convergent series in powers of α/l, where l is the length of cylinder.

16. Mr. Wright pointed out that the basis of a correspondence between two sets of geometric loci must lie in the theory of continuous groups, and he gives some results of a general nature. As a particular application he considered correspondences arising out of the projective group taken together with the straight line in space of three dimensions. He thus obtained theorems that certain conformal and projective groups are simply isomorphic. He also developed, from the group

point of view, the correspondence between three-dimensional lines and four-dimensional points, that between screws and hyperspheres, and that between five-dimensional points and three-dimensional screws.

17. If a family of surfaces $u(x, y, z) = \text{const.}$ can form part of a triply orthogonal system, they must satisfy a differential equation of the third order, $\Delta = 0$. Δ must be a differential invariant, and hence it is an algebraic invariant of certain known forms. Mr. Wright shows that it is an algebraic invariant of one linear and three quadratic forms in three variables, and its vanishing is equivalent to the geometric condition that the line meet the conics in six points in involution.

18. The formula

$$\frac{2u}{\pi}\int_0^1 dt \int_0^1 \frac{F'(\lambda u t)\lambda d\lambda}{\sqrt{1-\lambda^2}\sqrt{1-t^2}} = F(u) - F(0),$$

first given by Abel, has hitherto been proved only under quite narrow restrictions. Miss Smith's proof founded on the properties of double integrals is valid under very broad conditions. The formula is applied to an investigation of the condition under which the development of an arbitrary function in the form

$$f(x) = \sum_{u=0}^{u=\infty} a_u J_0(ux)$$

is valid. This development, outlined by Schlömilch in 1857, seems, up to the present, to lack any rigorous demonstration of its validity, even under conditions far more stringent than here required.

19. Professor Bjerknes showed how the extensive analogy existing between certain hydrodynamic phenomena and those of electric or magnetic fields could be developed from elementary principles, the principle of continuity and the principle of kinetic buoyancy. This last principle as well as the results relating to the analogy just mentioned were illustrated by a series of experiments.

20. Dr. Stephens's paper will be published in full in the *Transactions*.

21. Professor Pupin's paper discusses the quasi-discontinuous variations of the electric current in a sectional conductor when a steady electromotive force is impressed upon it. Thomson's mathematical discussion of the electric cable problem in 1855 and Kirchhoff's solution in 1858 of Thomson's problem in more general form are special cases of that here treated. The paper contributes an addition to the exceedingly small number of existing illustrations of the application of the Lagrangian method of solving a certain class of differential equations by a finite series of harmonics, the extension of which led to the discovery of Fourier's series. The solution discussed in the paper seems to offer a field for physical applications.

22. Dr. Quinn presented two preliminary theorems on properties of the cissoid, which suggest a mode of constructing a linkage for its kinematic description.

F. N. COLE,
Secretary.

NOTE ON CERTAIN GROUPS OF TRANSFORMATIONS OF THE PLANE INTO ITSELF.

BY DR. PETER FIELD.

(Read before the American Mathematical Society, December 29, 1905.)

IN the study of a plane curve or other configuration determined by five points,* four of the points may be taken as fixed and all the curves can be obtained by taking the fifth as any point in a given one of the one hundred and twenty regions determined in Professor Slaught's thesis.† Professor Slaught's diagram applies to the case of five real points. It is the purpose of this note to indicate diagrams corresponding to the case when four or two of the points are imaginary.

* Del Pezzo, *Rendiconti Accad. scienze fisiche matem.*, Napoli, ser. 2, vol. 3 (1889), pp. 46–49. Field, *Transactions Amer. Math., Society*, January, 1906.

† "The cross ratio group of one hundred and twenty quadratic Cremona transformations of the plane," *Amer. Journ. of Math.*, vols. 22 and 23. Also see Moore, "The cross ratio group of $n!$ Cremona transformations of order $n-3$ in flat space of $n-3$ dimensions," *Amer. Journ. of Math.*, vol. 22, No. 3, pp. 279–291, and Kantor, Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene (Berlin, 1895).

1. *Four Imaginary Points.*

Let the points be $a \equiv (i: 0: 1)$, $b \equiv (-i: 0: 1)$, $c \equiv (0: i: 1)$, $d \equiv (0: -i: 1)$, the fifth point being any real point $x_1: y_1: z_1$. It is clear that the fifth point can not be projected into either of the first four, while the different ways in which the first four can be projected into themselves are given by the transitive group of order 8 and degree 4 [the generators of the group might be taken as the substitutions $(adbc)$ and (ab)]. If $z=0$ be the line at infinity, the fifth point may be taken as any point in one of the eight regions bounded by the lines $x = 0$, $y = 0$, $x - y = 0$, $x + y = 0$. This is evident when it is noticed that the generators of the group of transformations corresponding to the preceding substitution are $x' : y' : z' = y_1 : -x_1 : z_1$ and $x' : y' : z' = -x_1 : y_1 : z_1$.

2. *Two Imaginary Points.*

In case but two of the points are imaginary let the five points be taken as follows: $g_1 \equiv (0:0:1)$, $h_1 \equiv (1:1:1)$, $c \equiv (i:1:0)$, $d \equiv (-i:1:0)$, $e \equiv (x_1 : y_1 : z_1)$. Four of the preceding five points can be projected into the first four in twelve different ways, because any one of the three real points may be projected into g_1 and either of the two remaining into h_1, while c and d may remain invariant or be interchanged. To project (g_1, h_1, d, e), (g_1, e_1, c, d), (h_1, g_1, c, d) into (g_1, h_1, c, d) we have the transformations

$$x' : y' : z' = y : x : z, \qquad x' : y' : z' = (x_1 + y_1)x - (x_1 - y_1)y :$$

$$(x_1 - y_1)x + (x_1 + y_1)y : \frac{x_1^2 + y_1^2}{z_1} z, \quad x' : y' : z' = z - x : z - y : z.$$

For these three cases the coördinates of the new fifth point in terms of x_1, y_1, z_1, are respectively

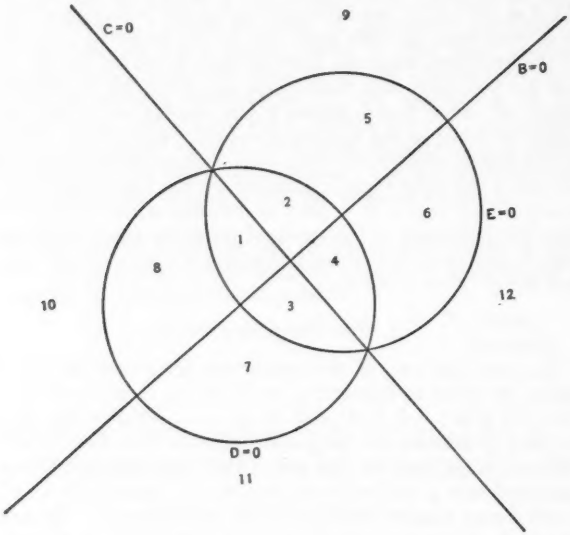
$$x' : y' : z' = y_1 : x_1 : z_1, \tag{1}$$

$$x' : y' : z' = 2y_1z_1 : 2x_1z_1 : x_1^2 + y_1^2, \tag{2}$$

$$x' : y' : z' = z_1 - x_1 : z_1 - y_1 : z_1. \tag{3}$$

These last transformations generate a group of order 12. The boundaries of the regions are the curves $A \equiv z = 0$,

$B \equiv x - y = 0$, $C \equiv x + y - z = 0$, $D \equiv x^2 + y^2 - 2z^2 = 0$, $E \equiv (x-z)^2 + (y-z)^2 - 2z^2 = 0$. The way in which the



regions are permuted under the transformations (1), (2), (3) is shown in the following table:

(1)	(13), (24), (56), (78), (9 12), (10 11),
(2)	(1 12), (26), (39), (45), (7 10), (8 11),
(3)	(14), (23), (57), (68), (9 11), (10 12),

In the figure above the line A is taken as the line at infinity.

UNIVERSITY OF MICHIGAN,
November, 1905.

THE MERAN MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held, in affiliation with the seventy-seventh regular convention of the Deutsche Naturforscher und Aerzte, at Meran, in the Tyrol, September 24–30, 1905. Eighty persons attended the sessions, which were presided over by Professors Köhler, Stäckel, Czuber, Geiser and Hensel respectively. The following papers were read:

(1) Professor CZUBER, Vienna: "The question of introducing the calculus into the preparatory schools, from an Austrian standpoint (Report)."

(2) Professor HOČEVAR, Graz: "Should the calculus be introduced into the preparatory schools, or not?"

(3) Professor WĀLSCH, Brünn: "Geometric-mechanic pictures of a new binary-invariant writing of chemical formulas."

(4) Professor HERZ, Vienna: "The year of Christ's birth."

(5) Professor WIEN, Würzburg: "Report on the partial differential equations of physics."

(6) Professor HASENÖHRL, Vienna: "On methods of integration of Maxwell equations for electric vibrations."

(7) Professor GANS, Tübingen: "Gravitation and electromagnetism."

(8) Professor WIRTINGER, Vienna: "On branching of functions of two variables."

(9) Dr. KOEBE, Berlin: "On conformal representation of multiply connected regions, particularly of regions that are bounded by arcs of circles."

(10) Professor LEVI-CIVITA, Padua: "On a technical problem in relation to conformal representation."

(11) Professor SCHOENFLIES, Königsberg: "On the so-called paradoxes of the theory of aggregates."

(12) Professor HENSEL, Marburg: "On the arithmetic properties of algebraic and of transcendental numbers."

(13) Professor ZINDLER, Innsbruck: "Report on the development of differential line geometry."

(14) Professor EPSTEIN, Strassburg: "Report on new investigations of the Riemann ζ-function and its extensions."

(15) Professor KOHN, Vienna: "On doubly apolar quadrics."

(16) Professor SCHLESINGER, Clausenburg: "On a representation of the system of absolute geometry."

(17) Dr. GRÜNWALD, Vienna : "Certain geometric applications of dual numbers."

(18) Professor MÜLLER, Vienna : "Contributions to cyclography."

(19) Professor MÜLLER, Vienna : "Descriptive geometry as an illustration of abstract projective geometry."

(20) Professor DOKULIL, Vienna : "Photogrammetry in the service of the history of art."

(21) Professor DOKULIL, Vienna : "Construction and study of stereoscopic pictures."

(22) Dr. FEJÉR, Clausenburg : "On equilibrium in a resisting medium."

(23) Professor STÄCKEL, Hanover : "Pairs of isometric surfaces."

Abstracts of the papers, so far as obtainable, follow. They are numbered to correspond to the titles in the list above.

1. Austrian schools give more time to mathematics and the sciences than is allowed in Germany, but Professor Czuber urges that the principles of the theory of functions be introduced empirically much earlier than at present. The idea of a derivative and of a definite integral could be taught in the gymnasia without lengthening the time devoted to mathematics. In conclusion he stated exactly formulated proposals as to the manner of carrying out the reform. The hearers, with one exception, agreed heartily with Professor Czuber's views.

2. Professor Hoĉevar answers his question in the affirmative and further discusses the plan outlined in 1.

5. This report will appear in full in the *Jahresbericht*.

6. Professor Hasenöhrl's paper shows how to apply certain methods of the theory of potential and related subjects to the integration of Maxwell's equations.

7. Professor Gans developed the Lorentz idea, that electrons of the same sign repel each other rather less strongly than those of opposite sign attract each other. A conductor may give no insulation for gravitation, yet furnish complete insulation for electrostatic forces. The deviation of this law of attraction from Newton's law depends only on quantities of the second

order. The consequences of this deviation should be investigated.

8. After giving a detailed discussion of the branching of the function $z^3 + 3xz + 2y = 0$ at $x = 0$, $y = 0$, Professor Wirtinger offered various considerations of the branching of a Riemann space, and interpreted the results by analysis situs.

9. Dr. Koebe proved the following theorem: If τ, τ' be two smooth plane regions, each bounded by $\rho + 1$ circles, of which no two cut or touch each other, and if τ can be mapped conformally upon τ', then the analytic function by means of which the representation is effected must be an integral or fractional linear function, and extended the result to apply to certain Riemann surfaces.

10. A problem in technical electricity was shown to depend upon the conformal representation of a multiply connected region bounded by circles. That discussed by Professor Levi-Civita was the determination of the most advantageous arrangement of the auxiliary conductor in the Wurtz lightning conductor.

11. This paper is to appear in the *Jahresbericht.*

13. The full report of which the present paper forms a part will be published in the *Jahresbericht.*

14. After speaking of the properties of Riemann's zeta functions, Dr. Epstein divided the generalizations into two groups, first, that of Dedekind, investigated by Landau, which relates to algebraic numbers, and second that studied by Lipschitz, Hurwitz and others, which bears the same relation to Riemann's zeta function as theta functions of p variables do to elliptic theta functions.

15. Professor Kohn showed that two quadric surfaces would be doubly apolar when one contains two generators of the same system which are polar reciprocal as to the other. Each surface has then a pair of lines reciprocal as to the other, in each generation. The quadrilaterals formed by these generators were then investigated.

16. The subject matter of Professor Schlesinger's paper will be incorporated in his forthcoming book on absolute geometry.

17. Professor Grünwald applies the numbers $\mu + \epsilon\nu\,(\epsilon^2 = 0)$ to the study of directed lines in space.

23. From every pair of applicable surfaces it is possible to derive twenty-eight pairs of applicable surfaces. Professor Stäckel's paper appeared in full in the *Jahresbericht*, volume 14, pages 507–516.

E. A. Miller,
Elijah Swift.

THE PRESENT AND THE FUTURE OF MATHEMATICAL PHYSICS.*

ADDRESS DELIVERED BEFORE THE SECTION OF APPLIED MATHEMATICS OF THE INTERNATIONAL CONGRESS OF ARTS AND SCIENCE, ST. LOUIS, SEPTEMBER 24, 1904.

BY PROFESSOR H. POINCARÉ.

What is the present state of mathematical physics? What are its problems? What is its future? Is it about to change its orientation? Will the object and methods of this science appear in ten years to our immediate successors in the same light as they appear to us? Or are we to witness a far-reaching transformation? These are the questions we are forced to face to-day at the outset of our inquiry.

It is easy to ask; difficult to answer. If we felt tempted to hazard a prediction, we should easily resist this temptation by stopping to think of the nonsense the most eminent scholars of a hundred years ago would have spoken in answer to the question of what this science would be in the nineteenth century. They would have thought themselves bold in their predictions; and after the event how timid we should have found them! Do not expect of me therefore any kind of prophesy.

But if, like all prudent physicians, I refuse to give a prognosis, still I cannot deny myself a little diagnosis. Well, then, yes; there are symptoms of a serious crisis, which would seem

* Translated with the author's permission by Professor J. W. Young.

to indicate that we may expect presently a transformation. However, there is no cause for great anxiety. We are assured that the patient will not die, and indeed we may hope that this crisis will be salutary, since the history of the past would seem to insure that. In fact, this crisis is not the first, and in order to understand it it is well to recall those which have gone before. Allow me a brief historical sketch.

Mathematical physics, as we are well aware, is an offspring of celestial mechanics, which gave it birth at the end of the eighteenth century at the moment when it had itself attained its complete development. The child, especially during its first years, showed a striking resemblance to its mother.

The astronomical universe consists of masses, undoubtedly of great magnitude, but separated by such immense distances that they appear to us as material points; these points attract each other in the inverse ratio of the squares of their distances, and this attraction is the only force which affects their motion. But if our senses were keen enough to show us all the details of the bodies which the physicist studies, the spectacle thus disclosed would hardly differ from the one which the astronomer contemplates. There too we should see material points separated by intervals which are enormous in comparison with their dimensions, and describing orbits according to regular laws. These infinitesimal stars are the atoms. Like the stars proper, they attract each other or repel, and this attraction or repulsion, which is along the line joining them, depends only on the distance. The law according to which this force varies with the distance is perhaps not the law of Newton, but it is analogous thereto: instead of the exponent -2 we probably have another exponent, and from this diversity in the exponents proceeds all the diversity of the physical phenomena, the variety in qualities and sensations, all the world of color and sound which surrounds us; in a word, all nature.

Such is the primitive conception in its utmost purity. Nothing remains but to inquire in the different cases, what value must be given to this exponent in order to account for all the facts. On this model, for example, Laplace constructed his beautiful theory of capillarity; he simply regards the latter as a special case of attraction, or, as he says, of universal gravitation, and no one is surprised to find it in the middle of one of the five volumes of his celestial mechanics. More recently Briot believes he has laid bare the last secret of optics, when he

has proved that the atoms of the ether attract each other in the inverse sixth power of the distance; and does not Maxwell, Maxwell himself, say somewhere that the atoms of a gas repel each other in the inverse ratio of the fifth power of the distance? We have the exponent -6 or -5, instead of the exponent -2; but it is always an exponent.

Among the theories of this period there is a single one that forms an exception, namely that of Fourier; here there are indeed atoms acting at a distance; they send each other heat, but they do not attract each other, they do not stir. From this point of view, Fourier's theory must have appeared imperfect and provisional to the eyes of his contemporaries, and even to his own.

This conception was not without greatness; it was alluring, and many of us have not definitely given it up; they know that the ultimate elements of things will not be attained, except by disentangling with patience the complex skein furnished us by our senses; that progress should be made step by step without neglecting any intermediate portions; that our fathers were unwise in not wishing to stop at all the stations; but they believe that when we once arrive at these ultimate elements, we shall meet again the majestic simplicity of celestial mechanics.

Nor has this conception been useless; it has rendered us a priceless service inasmuch as it has contributed to making more precise the fundamental concept of the physical law. Let me explain: What did the ancients understand by a law? It was to them an internal harmony, statical as it were, and unchangeable; or else a model which nature tried to imitate. To us a law is no longer that at all; it is a constant relation between the phenomenon of to-day and that of to-morrow; in a word, it is a differential equation.

Here we have the ideal form of the physical law; and, indeed, it is Newton's law which first gave it this form. If, later on, this form has become inured in physics, it has become so precisely by copying as far as possible this law of Newton, by using celestial mechanics as a model.

Nevertheless there came a day when the conception of central forces appeared no longer to suffice, and this is the first of the crises to which I referred a moment ago.

What was done? Abandoned was the thought of exploring the details of the universe, of isolating the parts of this vast mechanism, of analyzing one by one the forces which set them

going; and one was content to take as guides certain general principles which have precisely the object of relieving us of this minute study. How is this possible? Suppose we have before us any kind of machine; the part of the mechanism where the power is applied and the ultimate resultant motion alone are visible, while the transmissions, the intermediate gearing whereby the motion is communicated from one part to another, are hidden in the interior and escape our notice; we know not whether the transmissions are made by cog-wheels or by belts, by connecting-rods or other contrivances. Shall we say that it is impossible for us to learn anything about this machine unless we are allowed to take it apart? You well know that such is not the case, and that the principle of the conservation of energy suffices to furnish us the most interesting feature. We can easily show that the last wheel turns ten times more slowly than the first, since these two wheels are visible; and we can conclude therefrom that a couple applied to the first will be in equilibrium with a couple ten times as great applied to the second. To obtain this result, it is in no wise necessary to look into the mechanism of this equilibrium, or to know how the forces balance in the interior of the machine; it is sufficient to make sure that it is impossible for this balancing not to take place.

Very well! In the case of the universe, the principle of the conservation of energy can render us the same service. This universe also is a machine, much more complicated than any in use in the industries, of which nearly all the parts are deeply hidden; but by observing the motion of those which we can see, we can by the aid of this principle draw conclusions which will remain valid no matter what the details of the invisible mechanism which actuates them.

The principle of the conservation of energy, or Mayer's principle, is certainly the most important, but it is not the only one; there are others from which we can derive the same advantage. These are:

Carnot's principle, or the principle of the dissipation of energy.

Newton's principle, or the principle of the equality of action and reaction.

The principle of relativity, according to which the laws of physical phenomena must be the same for a stationary observer as for one carried along in a uniform motion of translation, so that we have no means, and can have none, of determining whether or no we are being carried along in such a motion.

The principle of the conservation of mass, or Lavoisier's principle. I will add the principle of least action.

The application of these five or six general principles to the various physical phenomena suffices to teach us what we may reasonably hope to know about them. The most remarkable example of this new mathematical physics is without doubt Maxwell's electro-magnetic theory of light. What is the ether? How are its molecules distributed? Do they attract or repel each other? Of these things we know nothing. But we know that this medium transmits both optical and electrical disturbances; we know that this transmission must take place in conformity with the general principles of mechanics and that suffices to establish the equations of the electro-magnetic field.

These principles are the boldly generalized results of experiment; but they appear to derive from their very generality a high degree of certainty. In fact, the greater the generality, the more frequent are the opportunities for verifying them, and such verifications, as they multiply, as they take the most varied and most unexpected forms, leave in the end no room for doubt.

Such is the second phase of the history of mathematical physics, and we have not yet left it. Shall we say that the first has been useless, that for fifty years science was on a wrong path and that there is nothing to do but to forget all that accumulation of effort which a vicious conception from the very beginning doomed to failure? By no means! Do you think the second period could have existed without the first? The hypothesis of central forces contained all the principles; it involved them as necessary consequences; it involved the principle of the conservation of energy, as well as that of mass, and the equality of action and reaction, and the law of least action, which appeared to be sure, not as experimental facts, but as theorems, and of which the statement had I know not how much greater precision and lesser generality than under their present form.

It is the mathematical physics of our fathers which has gradually made us familiar with these various principles, which has taught us to recognize them in the different garbs in which they are disguised. They have been compared with the results of experiment; it has been found necessary to change their expression in order to make them conform to the facts; thus they have been extended and strengthened. In this way they came to be regarded as experimental truths. The conception of central forces then became a useless support, or rather an

encumbrance, inasmuch as it imposed upon the principles its own hypothetical character.

The bounds then are not broken, because they were elastic; but they have been extended. Our fathers who established them have not labored in vain; and in the science of to-day we recognize the general features of the outline they traced.

Are we now about to enter upon a third period? Are we on the eve of a second crisis? Are these principles on which we have reared everything about to fall in their turn? This has recently become a vital question.

Hearing me speak thus, you are thinking without doubt of radium, that great revolutionary of the present day; and indeed I shall return to it presently. But there is something else. It is not merely the conservation of energy that is concerned; all the other principles are in equal danger, as we shall see by successively passing them in review.

Let us begin with Carnot's principle. It is the only one which does not present itself as an immediate consequence of the hypothesis of central forces; quite to the contrary, indeed, it appears, if not actually to contradict this hypothesis, at least not to be reconcilable with it without some effort. If physical phenomena were due exclusively to the motion of atoms the mutual attractions of which depend only on the distance, it would seem that all these phenomena should be reversible; if all the initial velocities were reversed, these atoms, if still subject to the same forces, should traverse their trajectories in the opposite direction, just as the earth would describe backward this same elliptical orbit that it now describes forward, if the initial conditions of its motion had been reversed. Thus, if a physical phenomenon is possible, the inverse phenomenon should be equally possible, and one should be able to retrace the course of time. Now, it is not so in nature, and this it is precisely that the principle of Carnot teaches us; heat may pass from a hot body to a cold; it is impossible to compel it to take the opposite route and to re-establish differences of temperature which have disappeared. Motion can be entirely destroyed and transformed into heat by friction; the converse transformation can only occur partially.

Efforts have been made to reconcile this apparent contradiction. If the world tends toward uniformity, it is not because its ultimate parts, though diversified at the start, tend to become less and less different; it is because moving at random

they become mixed. To an eye which could distinguish all the elements, the variety would remain always as great; every grain of this powder retains its originality and does not fashion itself after its neighbors; but as the mixture becomes more and more perfect, our rough senses perceive only uniformity. That is why, for example, temperatures tend to equalize themselves, without its being possible to go back.

A drop of wine, let us say, falls into a glass of water; whatever the internal motion of the liquid, we shall soon see it assume a uniformly roseate hue, and from then on no possible shaking of the vessel would seem to be capable of again separating the wine and the water. Here, then, we have what may be the type of the irreversible phenomenon of physics: to hide a grain of barley in a great mass of wheat would be easy; to find it again and to remove it is practically impossible. All this has been explained by Maxwell and Boltzmann, but the man who has put it most clearly was Gibbs, in a book too little read because it is a little difficult to read, in his Elements of Statistical Mechanics.

To those who take this point of view, Carnot's principle is an imperfect principle, a sort of concession to the frailty of our senses; it is because our eyes are too coarse that we do not distinguish the elements of the mixture; it is because our hands are too coarse that we can not compel them to separate; the imaginary demon of Maxwell, who can pick out the molecules one by one, would be quite able to constrain the world to move backwards. That it should return of its own accord is not impossible; it is only infinitely improbable; the chances are that we should wait a long time for that combination of circumstances which would permit a retrogression; but, sooner or later, they will occur, after years, the number of which would require millions of figures. These reservations, however, all remained theoretical; they caused little uneasiness and Carnot's principle preserved all of its practical value.

But now here is where the scene changes. The biologist, armed with his microscope, has for a long time noticed in his preparations certain irregular motions of small particles in suspension; this is known as Brown's motion. He believed at first that it was a phenomenon of life, but he soon saw that inanimate bodies hopped about with no less ardor than others; he then turned the matter over to the physicists. Unfortunately, the physicists did not become interested in the question for a long time.

Light is concentrated, so they argued, in order to illuminate the microscopical preparation; light involves heat, and this causes differences in temperature and these produce internal currents in the liquid, which bring about the motions referred to.

M. Gouy had the idea of looking a little more closely, and thought he saw that this explanation was untenable; that the motion becomes more active as the particles become smaller, but that they are uninfluenced by the manner of lighting. If, then, these motions do not cease, or, rather, if they come into existence incessantly, without borrowing from any external source of energy, what must we think? We must surely not abandon on this account the conservation of energy; but we see before our eyes motion transformed into heat by friction and conversely heat changing into motion, and all without any sort of loss, since the motion continues forever. It is the contradiction of Carnot's principle. If such is the case, we need no longer the infinitely keen eye of Maxwell's demon in order to see the world move backward; our microscope suffices. The larger bodies, those of a tenth of a millimeter, for example, are bombarded from all sides by the moving atoms, but they do not stir, because these shocks are so numerous that the law of probabilities requires them to compensate each other; but the smaller particles are hit too rarely to have this compensation take place with any degree of certainty and are thus incessantly tossed about. And so one of our principles is already in danger.

Let us consider the principle of relativity; this principle is not only confirmed by our daily experience, not only is it the necessary consequence of the hypothesis of central forces, but it appeals to our common sense with irresistible force. And yet it also is being fiercely attacked. Let us think of two electrified bodies; although they seem to be at rest, they are, both of them, carried along with the motion of the earth; Rowland has shown us that an electric charge in motion is equivalent to a current; these two charged bodies, then, are equivalent to two parallel currents in the same direction; these two currents should attract each other. By measuring this attraction we should be measuring the velocity of the earth; not its velocity relative to the sun and the stars, but its absolute velocity.

I know what will be said; it is not its absolute velocity; it is its velocity relative to the ether. But, how unsatisfactory that is! Is it not clear that with this interpretation, nothing could be inferred from the principle? It could no longer teach

us anything, simply because it would no longer fear any contradiction. Whenever we have succeeded in measuring anything, we would always be free to say that it is not the absolute velocity, and if it is not the velocity relative to the ether, it might always be the velocity relative to some new unknown fluid with which we might fill all space.

And then experiment, too, has taken upon itself to refute this interpretation of the principle of relativity; all the attempts to measure the velocity of the earth relative to the ether have led to negative results. Herein experimental physics has been more faithful to the principle than mathematical physics; the theorists would have dispensed with it readily in order to harmonize the other general points of view; but experimentation has insisted on confirming it. Methods were diversified; finally Michelson carried precision to its utmost limits; nothing came of it. It is precisely to overcome this stubborness that to-day mathematicians are forced to employ all their ingenuity.

Their task was not easy, and if Lorentz has succeeded, it is only by an accumulation of hypotheses.

The most ingenious idea is that of local time. Let us imagine two observers, who wish to regulate their watches by means of optical signals; they exchange signals, but as they know that the transmission of light is not instantaneous, they are careful to cross them. When station B sees the signal from station A, its timepiece should not mark the same hour as that of station A at the moment the signal was sent, but this hour increased by a constant representing the time of transmission. Let us suppose, for example, that station A sends its signal at the moment when its time-piece marks the hour zero, and that station B receives it when its time-piece marks the hour t. The watches will be set, if the time t is the time of transmission, and in order to verify it, station B in turn sends a signal at the instant when its time-piece is at zero; station A must then see it when its time-piece is at t. Then the watches are regulated.

And, indeed, they mark the same hour at the same physical instant, but under one condition, namely, that the two stations are stationary. Otherwise, the time of transmission will not be the same in the two directions, since the station A, for example, goes to meet the disturbance emanating from B, whereas station B flees before the disturbance emanating from A. Watches regulated in this way, therefore, will not mark the

true time; they will mark what might be called the local time, so that one will gain on the other. It matters little, since we have no means of perceiving it. All the phenomena which take place at A, for example, will be behind time, but all just the same amount, and the observer will not notice it since his watch is also behind time; thus, in accordance with the principle of relativity he will have no means of ascertaining whether he is at rest or in absolute motion. Unfortunately this is not sufficient; additional hypotheses are necessary. We must admit that the moving bodies undergo a uniform contraction in the direction of the motion. One of the diameters of the earth, for example, is shortened by $1/200000000$ as a result of our planet's motion, whereas the other diameter preserves its normal length. Thus we find the last minute differences accounted for. Then there is still the hypothesis concerning the forces. Forces, whatever their origin, weight as well as elasticity, will be reduced in a certain ratio in a world endowed with a uniform translatory motion; or rather that would happen for the components at right angles to the direction of translation; the parallel components will not change. Let us then return to our example of the two electrified bodies; they repel each other; but at the same time, if everything is carried along in a uniform translation, they are equivalent to two parallel currents in the same direction, which attract each other.

This electrodynamic attraction is, then, subtracted from the electrostatic repulsion, and the resultant repulsion is weaker than if the two bodies had been at rest. But since we must, in order to measure this repulsion, balance it by another force, and since all these other forces are reduced in the same ratio, we observe nothing. Everything, then, appears to be in order. But have all doubts been dissipated? What would happen if we could communicate by signals other than those of light, the velocity of propagation of which differed from that of light? If, after having regulated our watches by the optical method, we wished to verify the result by means of these new signals, we should observe discrepances due to the common translatory motion of the two stations. And are such signals inconceivable, if we take the view of Laplace, that universal gravitation is transmitted with a velocity a million times as great as that of light?

Thus the principle of relativity has in recent times been valiantly defended; but the very vigor of the defense shows how serious was the attack.

And now let us speak of the principle of Newton, concerning the equality of action and reaction. This principle is intimately connected with the preceding and it would seem that the fall of one would involve the fall of the other. Nor must we be surprised to find here again the same difficulties.

The electrical phenomena, it is thought, are due to displacements of small charged particles called electrons which are immersed in the medium we call ether. The motions of these electrons produce disturbances in the surrounding ether; these disturbances are propagated in all directions with the velocity of light, and other electrons initially at rest are displaced when the disturbance reaches the portions of the ether in which they lie. The electrons, then, act one upon the other, but this action is not direct; it takes place by mediation of the ether. Under these conditions, is it possible to have equality between action and reaction, at least for an observer who takes account only of the motion of matter, that is of the electrons, and who ignores that of the ether which he is unable to see? Evidently not. Even if the compensation were exact, it could not be instantaneous. The disturbance is propagated with a finite velocity; it reaches the second electron, therefore, only after the first has long been reduced to rest. This second electron will, then, after an interval, be subjected to the action of the first, but will certainly not at that moment react upon it, since there is no longer anything in the neighborhood of this first electron that stirs.

The analysis of the facts will allow us to become more definite. Let us imagine, for example, a Hertzian oscillator such as those used in wireless telegraphy; it sends energy in all directions; but we may attach to it a parabolic mirror, as was done by Hertz with his smallest oscillators, so as to send all the energy produced in a single direction. What then will happen according to the theory? Why, the apparatus will recoil as though it were a cannon and the projected energy a ball, and that contradicts the principle of Newton, since our present projectile has no mass; it is not matter, it is energy. It is the same, moreover, in the case of a light-house having a reflector, since light is merely a disturbance in the electro-magnetic field. This light-house would recoil, as though the light it sends forth were a projectile. What is the force that must produce this recoil? It is what is known as the Maxwell-Bartholdi pressure; it is very small, and to put it in evidence caused much trouble,

even with the most sensitive radiometers; but it is sufficient for our purpose that it exists.

If all the energy issuing from our oscillator strikes a receiver, the latter will act as though it had received a physical shock, which in a sense will represent the compensation of the oscillator's recoil; the reaction will be equal to the action, but they will not be simultaneous; the receiver will advance, but not at the instant when the oscillator recoils. If the energy is propagated indefinitely without meeting a receiver, the compensation will never take place.

Shall we say that the space which separates the oscillator from the receiver and which the disturbance must traverse in passing from one to the other, is not empty, but is filled not only with ether, but with air, or even in inter-planetary space with some subtile, yet ponderable fluid; that this matter receives the shock, as does the receiver, at the moment the energy reaches it, and recoils, when the disturbance leaves it? That would save Newton's principle, but it is not true. If the energy during its propagation remained always attached to some material substratum, this matter would carry the light along with it and Fizeau has shown, at least for the air, that there is nothing of the kind. Michelson and Morley have since confirmed this. We might also suppose that the motions of matter proper were exactly compensated by those of the ether; but that would lead us to the same considerations as those made a moment ago. The principle, if thus interpreted, could explain anything, since whatever the visible motions we could imagine hypothetical motions to compensate them. But if it can explain anything, it will allow us to foretell nothing; it will not allow us to choose between the various possible hypotheses, since it explains everything in advance. It therefore becomes useless.

And then the suppositions that must be made concerning the motions of the ether are not very satisfactory. If the electric charges were doubled, it would be natural to suppose that the velocities of the atoms of the ether also became twice as great, and for the compensation it would be necessary that the mean velocity of the ether become four times as great.

This is why I have for a long time thought that these consequences of the theory, which contradict Newton's principle, would some day be abandoned; and yet the recent experiments on the motion of the electrons emitted by radium seem rather to confirm them.

I now come to Lavoisier's principle concerning the conservation of mass. This is certainly a principle which cannot be tampered with without shaking the science of mechanics. And still there are persons who think that it seems true to us only because in mechanics we consider only moderate velocities, and that it would cease to be so for bodies having velocities comparable with that of light. Now, such velocities are at present believed to have been realized; the cathode rays and those of radium would seem to be formed of very minute particles or electrons that move with velocities that are no doubt less than that of light, but which appear to be about one tenth or one third of it.

These rays can be deflected either by an electric or by a magnetic field, and by comparing these deflections it is possible to measure both the velocity of the electrons and their mass (or rather the ratio of their mass to their charge). But it was found that as soon as these velocities approached that of light a correction was necessary. Since these particles are electrified, they cannot be displaced without disturbing the ether; to put them in motion, it is necessary to overcome a double inertia, that of the particle itself and that of the ether. The total or apparent mass that is measured is then composed of two parts: the real or mechanical mass of the particle and the electrodynamic mass representing the inertia of the ether.

Now, the calculations of Abraham and the experiments of Kaufmann have shown that the mechanical mass properly so called is nothing, and that the mass of the electrons, at least of the negative electrons, is purely of electrodynamic origin. This is what compels us to change our definition of mass; we can no longer distinguish between the mechanical mass and the electrodynamic mass, because then the first would have to vanish; there is no other mass than the electrodynamic inertia; but in this case, the mass can no longer be constant; it increases with the velocity; and indeed it depends on the direction, and a body having a considerable velocity will not oppose the same inertia to forces tending to turn it off its path that it opposes to those tending to accelerate or retard its motion.

There is indeed another resource: the ultimate elements of bodies are electrons, some with a negative charge, others with a positive charge. It is understood that the negative electrons have no mass; but the positive electrons, from what little is known of them, would seem to be much larger. They perhaps

have besides their electrodynamic mass a true mechanical mass. The real mass of a body would then be the sum of the mechanical masses of its positive electrons, the negative electrons would not count; the mass defined in this way might still be constant.

Alas, this resource is also denied. Let us recall what we said concerning the principle of relativity and the efforts made to save it. And it is not only a principle that is to be saved; the indubitable results of Michelson's experiments are involved. And so, as was above seen, Lorentz, to account for these results, was obliged to suppose that all forces, whatever their origin, are reduced in the same ratio in a medium having a uniform translatory motion. But that is not sufficient; it is not enough that this should take place for the real forces, it must also be the same in the case of the forces of inertia; it is necessary, therefore — so he says — that *the masses of all particles be influenced by a translation in the same degree as the electromagnetic masses of the electrons.*

Hence, the mechanical masses must vary according to the same laws as the electrodynamic; they can then not be constant.

Do I need to remark that the fall of Lavoisier's principle carries with it that of Newton's? The latter implies that the center of gravity of an isolated system moves in a straight line; but if there no longer exists a constant mass, there no longer exists a center of gravity; indeed the phrase would be meaningless. This is why I said above that the experiments on cathode rays seemed to justify the doubts of Lorentz concerning Newton's principle.

From all these results, if they were to be confirmed, would issue a wholly new mechanics which would be characterized above all by this fact, that there could be no velocity greater than that of light,* any more than a temperature below that of absolute zero. For an observer, participating himself in a motion of translation of which he has no suspicion, no apparent velocity could surpass that of light, and this would be a contradiction, unless one recalls the fact that this observer does not use the same sort of timepiece as that used by a stationary observer, but rather a watch giving the "local time."

* Because bodies would oppose an increasing inertia to the causes that would tend to accelerate their motion; and when approaching the velocity of light, this inertia would become infinite.

Here we are then face to face with a question, of which I shall confine myself to the mere statement. If there is no longer any mass what becomes of Newton's law?

Mass has two aspects: it is at the same time a coefficient of inertia and an attracting mass entering as a factor into Newton's law of attraction. If the coefficient of inertia is not constant, can the attracting mass be constant? This is the question.

The principle of the conservation of energy at least still remained and appeared more finely established. Shall I recall to your minds how it too was thrown into discredit? That event made more noise than the preceding; the journals are full of it. Ever since the first work of Becquerel, and above all after the Curies had discovered radium, it was seen that every radioactive substance was an inexhaustible source of radiation. Its activity seemed to continue without change through months and years. That is already a strain on the principles; these radiations in fact were energy, and from the same piece of radium came forth this energy and it came forth indefinitely. But these quantities of energy were too minute to be measured; at least that was the belief, and the matter caused little uneasiness.

The scene changed when Curie thought of placing the radium in a calorimeter. It was then seen that the quantity of heat continuously generated was very considerable.

The explanations advanced were numerous; but in a case of this kind it is not possible to say that an abundance of good does no harm: as long as one explanation has not displaced the others we can not be sure that any one of them is good. For some time, however, one of these explanations seems to be gaining the upper hand and we may reasonably hope that we hold the key to the mystery.

Sir W. Ramsey has attempted to show that radium is transformed, that it contains an enormous amount of energy, but not an inexhaustible amount. The transformation of radium must then produce a million times as much heat as any known transformation; the radium would be exhausted in 1250 years; that is not long, but you see that we are at least sure of being bound to the present state of affairs for some hundreds of years. While we wait our doubts subsist.

In the midst of such ruin, what remains standing? The principle of least action up to now is intact, and Larmor appears to think that it will long survive the others. It is in fact more vague and even more general.

In the presence of this general collapse of principles, what attitude should mathematical physics take? First of all, before becoming too excited, it is well to ask whether all this is really true. All this disparagement of principles is encountered only in the case of the infinitely small; the microscope is needed to see Brown's motion, the electrons are rather tiny, radium is very rare and never more than a few milligrams are together; and then we can ask whether by the side of the minute thing that was observed, there was not another minute thing which was not noticed and which counterbalanced the first.

The question is surely debatable, and apparently only experiment can solve it. We should merely have to turn the matter over to the experimenters and, while waiting for them definitely to settle the controversy, not to trouble ourselves with these disquieting problems, and to keep quietly at our work, as though the principles were still unchallenged. We certainly have enough to do without leaving the domain where they can be applied with all certainty; we have enough to keep us busy during this period of doubt.

And yet is it really true that we can do nothing to relieve science of these doubts? It must indeed be said that it has not been experimental physics alone that has brought them into existence; mathematical physics has contributed its share. It was the experimenters who saw radium emit energy; but the theorists were the ones who brought to light all the difficulties inherent in the propagation of light through a moving medium; had it not been for them, they probably would not have been noticed. They have, then, done their best to embarrass us; it is no more than just that they should help us to extricate ourselves.

They must subject to a searching criticism all the new conceptions that I have outlined to-day, nor must they abandon the principles except after a loyal effort to save them. What can they do in this direction? That is what I shall seek to explain.

Among the most interesting problems of mathematical physics a place should be set apart for those that belong to the kinetic theory of gases.

Much has already been done toward their solution, but much remains to do. This theory is an everlasting paradox. We have reversibility in the premises and irreversibility in the conclusions, and a deep chasm between the two. Will statistical

considerations, and the law of large numbers, suffice to fill it up? Many points still remain obscure to which it will be necessary to return, and that without doubt several times. In clearing them up, the meaning of Carnot's principle will be better understood, and its general position in dynamics; and we shall be better able to interpret the curious experiment of Gouy to which I referred above.

Should we not also make an effort to obtain a more satisfactory theory of the electrodynamics of moving bodies? It is here, above all, as I indicated sufficiently a short time ago, that the difficulties accumulate; even though we heap up hypotheses, we can not satisfy all the principles at once; no one has succeeded so far in saving some without sacrificing others. But all hope of obtaining better results is not yet lost. Let us, then, take the theory of Lorentz. Let us turn it over and over, let us modify it little by little, and all will be well, perhaps.

Indeed, instead of supposing that bodies in motion undergo a contraction in the direction of motion and that this contraction is the same whatever the nature of these bodies and the forces to which they are subjected, could not a simpler and more natural hypothesis be made? One might suppose, for example, that it is the ether which changes when it is in relative motion with respect to the material substance which passes through it; that, when thus modified, it no longer transmits the disturbances with the same velocity in all directions. It would transmit more rapidly those disturbances which are being propagated parallel to the motion of the substance, be it in the same direction or in the opposite, and less rapidly those which are propagated at right angles. The wave surfaces would then no longer be spheres, but ellipsoids, and one could do without this extraordinary contraction of all bodies.

I am giving this only by way of example, since the modifications which could be tried are evidently susceptible of infinite variation.

It is possible also that astronomy may some day furnish us with data on this point: she it was, in fact, who raised the question by making known to us the phenomenon of the aberration of light. If the theory of the aberration of light is roughly constructed, a curious result is arrived at. The apparent positions of the stars differ from their real positions by reason of the earth's motion, and since this motion is variable the apparent positions vary. The real position we are unable

to ascertain, but we can observe the variations of the apparent position. Observations on aberration then show us, not the earth's motion, but the variations of this motion. They can, therefore, teach us nothing concerning the absolute motion of the earth.

These at any rate are the facts under a first approximation; but such would no longer be the case if we could observe the thousandth part of a second. It would then be seen that the variation in the apparent motion of the star depends not only on the variation in the earth's motion, a variation which is well known, since it is the motion of our globe in its elliptical orbit, but also on the mean value of this motion, so that the constant of aberration would not be quite the same for all stars, and that the differences would make known to us the earth's absolute motion in space.

This would be, in another form, the end of the principle of relativity. We are far, it is true, from being able to observe the thousandth part of a second, but after all, say some, the earth's total absolute velocity is perhaps much greater than its velocity relative to the sun; if it were for instance 300 km. per second, instead of 30 km., that would suffice to make the phenomenon observable.

I believe that by reasoning in this manner we carry simplicity in the theory of aberration too far. Michelson has shown, as I have said, that the methods of physics are powerless to put absolute motion in evidence; I am convinced that in the case of astronomical methods it will be the same, no matter how far precision may be carried.

However that may be, the data which astronomy will furnish in this direction will one day be valuable to the physicist. In the meantime I believe that the theorists, keeping in mind the experiments of Michelson, may count on a negative result, and that they would do useful work by constructing a theory of aberration which takes account of it in advance.

But let us return to the earth. There, too, we can help the experimenters. We can, for example, prepare the way by studying thoroughly the dynamics of the electrons; not, be it well understood, by starting from a single hypothesis, but by multiplying the hypotheses as much as possible. It would then be the part of the physicist to use our work in searching for the crucial experiment which would decide between them.

This dynamics of the electrons can be approached from many

sides; but among the roads that lead there, there is one which has been somewhat neglected, and yet it is one of those that promise us the most surprises. It is the motion of electrons that produces the lines of the spectrum; this is proved by the phenomenon of Zeemann; what vibrates in an incandescent body is affected by a magnet, and is hence electrified. This is a first very important point; but no one has gone into the question any further. Why are the lines of the spectrum distributed according to a regular law? These laws have been studied by the experimenters in the greatest detail; they are very precise and comparatively simple. A first study of these arrangements brings to mind the harmonics encountered in acoustics; but the difference is great. Not only are the numbers of vibrations not successive multiples of a single number, but we even find nothing analogous to the roots of those transcendental equations, to which we are led by so many problems of mathematical physics: that of the vibrations of an elastic body of any shape, of the Hertzian oscillations in a generator of any form, the problem of Fourier on the cooling of a solid body.

The laws are simpler, but they are of an entirely different kind. To mention only one of these differences, for harmonics of high order the number of vibrations tends toward a finite limit, instead of increasing indefinitely.

This has not yet been explained, and I believe that here is one of the most important of nature's secrets. Lindemann has made a praiseworthy attempt, but in my opinion without success. This attempt should be renewed. We shall thus penetrate, so to speak, into the intimacies of matter. And, from the particular point of view that we occupy to-day, when we shall know why the vibrations of incandescent bodies differ in this way from the vibrations of ordinary elastic bodies, why the electrons do not behave like the matter with which we are familiar, we shall better understand the dynamics of the electrons and it will perhaps be easier for us to reconcile them with the principles.

Now suppose that all these efforts should fail (and when all is said, I do not believe they will), what should be done? Should we seek to rebuild these shattered principles by one stroke, as it were? That, evidently, is always possible, and I take back nothing of what I once said. "Did you not write," you might say, if you were seeking a quarrel with me, "did you not write

that the principles, though they are of experimental origin, are now beyond the possibility of experimental attack, because they have become conventions? And now you come to tell us that the triumphs of the most recent experiments put these principles in danger."

Very well, I was right formerly, and I am not wrong to-day. I was right formerly, and what is taking place at present is another proof of it. Let us take, for example, the calorimetric experiment of Curie with radium. Is it possible to reconcile it with the principle of the conservation of energy? This has been attempted in many ways; but there is one among them to which I wish to call your attention; it is not the explanation which is tending to-day to prevail, but it is one of those that have been suggested. Radium is assumed to be only an intermediary, merely to store radiations of an unknown nature that fly through space in all directions, traversing all bodies except radium without being changed by this passage, and without exerting on them any action whatever. Radium alone can appropriate a little of their energy and then return it to us in various forms.

How useful this explanation is and how convenient! In the first place it is non-verifiable and hence irrefutable. Then, it can serve to account for any contradiction to Mayer's principle; it answers in advance not only the objection of Curie, but all other objections that the experimenters of the future may accumulate. This new and unknown energy could be used for anything.

That is exactly what I said, and by such means it is easy to show that our principle is safe from experimental attacks.

But then, what have we gained by this stroke? The principle is intact, but henceforth what is it good for? It enabled us to foresee that under such and such conditions we could count on a certain amount of energy; it imposed a limit; but now that there has been placed at our disposal this indefinite supply of new energy, we are no longer limited by anything; and, as I have also written, if a principle ceases to be productive, experiment, without contradicting it directly, would nevertheless condemn it.

That, then, is not what should be done. We should have to rebuild from the beginning. If we were driven to this necessity, we could easily console ourselves. We should not be obliged to conclude that science can never do aught but the

work of a Penelope, that it can only raise ephemeral structures which it is soon forced to demolish completely with its own hands.

As I have said, we have already passed through a similar crisis. I have shown you that in the second mathematical physics, that of general principles, one finds traces of the first, that of central forces; it will be the same if we are to know a third. Just so with the animal that casts its outer shell, that bursts the skin that has become too small and grows a new one; under the new covering can always be recognized the essential traits of the organism that survives.

In what direction we are going to expand we are unable to foresee. Perhaps it is the kinetic theory of gases that will forge ahead and serve as a model for the others. In that case, the facts that appeared simple to us at first will be nothing more than the resultants of a very large number of elementary facts which the laws of probability alone would induce to work toward the same end. A physical law would then assume an entirely new aspect; it would no longer be merely a differential equation, it would assume the character of a statistical law.

Perhaps too we shall have to construct an entirely new mechanics, which we can only just get a glimpse of, where, the inertia increasing with the velocity, the velocity of light would be a limit beyond which it would be impossible to go. The ordinary, simpler mechanics would remain a first approximation since it would be valid for velocities that are not too great, so that the old dynamics would be found in the new. We should have no reason to regret that we believed in the older principles, and indeed since the velocities that are too great for the old formulas will always be exceptional, the safest thing to do in practice would be to act as though we continued to believe in them. They are so useful that a place should be saved for them. To wish to banish them altogether would be to deprive oneself of a valuable weapon. I hasten to say, in closing, that we are not yet at that pass, and that nothing proves as yet that they will not come out of the fray victorious and intact.

SHORTER NOTICES.

Carl Gustav Jacob Jacobi. Festschrift zur Feier der hundertsten Wiederkehr seines Geburtstages. Mit einem Bildnis und dem Faksimile eines Briefes. By LEO KÖNIGSBERGER. Leipzig, B. G. Teubner, 1904. 8vo. xvii + 554 pp.

IT is a pleasure as well as a duty for posterity to honor and keep green the memory of those great men of science whose discoveries have contributed to human happiness or whose profound meditations have opened up new and fruitful lines of research. To the latter class belongs Jacobi, the hundredth anniversity of whose birthday fell on December 10, 1904. The third International congress of mathematicians, held that year, determined to celebrate this event with due solemnity and Professor Königsberger was asked to prepare a commemorative address. No better choice of speaker could be made; for Professor Königsberger besides having published a noteworthy history of the elliptic functions for the years 1826–29 has contributed numerous important papers on the abelian transcendants, on differential equations and on principles of mechanics, all subjects which occupied Jacobi's unceasing attention.

The exceptional position that Jacobi occupies in the history of German mathematics makes the story of his life doubly interesting. At the time when he began his career the condition of mathematics in Germany was truly deplorable. The whole mathematical genius of the Teutonic people seems to have been absorbed by a single individual, Gauss, leaving a mere pittance to be thinly distributed among his contemporaries. Moreover the cold and reserved nature of Gauss, his uncommunicative if not secretive disposition, his unvarying practice of publishing nothing that was not perfect in form and complete in content, all contributed most unhappily in reducing to a minumum the mighty influence he should have exerted on his countrymen during the first quarter of the nineteenth century.

With the advent of Jacobi a new era began ; a new generation of mathematicians sprang up with a suddenness and brilliancy which is astounding. Jacobi was presently joined by Steiner, Dirichlet, Plücker and others, and Germany began the triumphal march which soon led her to the first rank in the world of mathematics.

The most prominent figure in this new generation is Jacobi.

His epoch-making discoveries in the theory of elliptic functions when only twenty-three years of age excited the admiration of all Europe, while his many and profound researches rapidly rendered him one of the most celebrated mathematicians of his day. But Germany had had great mathematicians before; Leibniz, Euler, the Bernoullis, not to mention later ones. What distinguishes Jacobi from his predecessors is the fact that he was a great teacher. In this respect he was the very opposite of his great contemporary Gauss, who disliked to teach, and who was anything but inspiring. Jacobi's lectures were models of clearness; the enthusiasm of the teacher kindled a like enthusiasm in his pupils. He took the unprecedented step of making his more gifted students acquainted with his own unfinished investigations and stimulated them to attack problems suggested by these researches. The results were most gratifying, for ere long we see his pupils, Hesse, Goepel, Rosenhain, Richelot, and others, publishing important memoirs.

It is not our intention, however, to write a sketch of Jacobi's life, but rather to call attention to the interest and the lessons which it has for us. We feel sure that no reader will regret perusing Professor Königsberger's biography, which is obviously a labor of love and the result of long and patient research. A feature of the work deserves especial commendation, and is most unusual. All of Jacobi's principal papers are carefully analyzed and the growth and filiation of his mathematical ideas stand forth in bold relief. It has no doubt cost his biographer months of patient study to do this; but the results here given will prove of utmost service to future generations of mathematicians who seek a rapid orientation of the leading facts in any part of Jacobi's multitudinous and varied publications.

JAMES PIERPONT.

Uebungsbuch zum Studium der höheren Analysis. VON OSKAR SCHLÖMILCH. Erster Theil: *Aufgaben aus der Differentialrechnung.* Fifth edition, prepared by Dr. E. NAETSCH. Leipzig, B. G. Teubner, 1904. 8vo. viii + 372 pp.

THIS work whose first edition appeared more than thirty years ago has enjoyed a well-merited popularity, both on account of the variety and careful choice of its problems and also the elegance and instructiveness often exhibited in their solutions. Numerous additions have been made in this, the fifth edition, giving fresh interest to the work. The problems

relating to curves and surfaces, maxima and minima, limits and infinite series, are especially praiseworthy.

JAMES PIERPONT.

A Course in Mathematical Analysis, by EDOUARD GOURSAT, Professor of Mathematics in the University of Paris; translated by EARLE RAYMOND HEDRICK, Professor of Mathematics in the University of Missouri. Volume I. Ginn and Company, Boston, 1905. 8vo. viii + 548 pp.

THE French edition of this work was published in 1902, and it was reviewed in the BULLETIN.* While it is true that advanced students of mathematics recognize the necessity of learning to read mathematical French and German, and equip themselves duly in this respect, the undergraduate finds the additional difficulty of a foreign language a serious handicap in the use of a mathematical text-book. And yet it is precisely for the undergraduate, the student in the second course in calculus and the first course in the theory of functions, who is perhaps preparing to specialize in applied mathematics and will not carry his study of analysis beyond the undergraduate courses, that Professor Goursat's book contains so much which is important but at present is not to be found in English text-books. Professor Hedrick has prepared the translation with great care and has made it a worthy reproduction of this standard work. In his preface he says: "Few alterations have been made from the French text. Slight changes of notation have been introduced occasionally for convenience, and several changes and additions have been made at the suggestion of Professor Goursat, who has very kindly interested himself in the work of translation." To the publishers is due much credit for the excellent typography of the book. Niceties of spacing and arrangement of the formulas, which hitherto have usually been neglected by American and English printers, here contribute to make the page extremely attractive.

WM. F. OSGOOD.

Elementary Modern Geometry. Part I.: *Experimental and Theoretical, Triangles and Parallels.* By H. G. WILLIS. Oxford, Clarendon Press, 1905. v + 236 pp.

THE order of sequence of the adjectives in the above title might lead to a misunderstanding; the subjects treated are those

*Cf. BULLETIN, ser. 2, vol. 9 (1902-03), p. 547.

of a beginner's school geometry, but the method of treatment is rather unusual. Indeed, so many text-books on geometry are being published that the appearance of still another seems hardly justified, unless it presents something new. The first seventy pages are empirical and experimental to develop space intuition without any attempt at formal demonstration. Paper folding plays an important part and most of the constructions of ordinary geometry are actually made. Free use is made of ruler, compasses and superposition.

After this somewhat extensive introduction, formal definitions, postulates and axioms are given, the latter including an axiom of continuity, an axiom of displacement and an axiom of rotation. The last axiom replaces Euclid's fifth postulate and is stated as follows: "If a straight line, after turning in a plane about various points in itself, coincides with its initial position, the algebraic sum of the angles turned through is zero or one or more straight angles." The definitions of a straight line and of a plane are those of Leibniz.

Demonstrations based on unexplained constructions are frequent in the book, even the possibility of the construction not being expressly postulated. The third chapter considers all these constructions and adds a good discussion of graphs, including the circle, hyperbola, sine curve (without trigonometry) and a long list of others as exercises, a good sample being

$$y^2 = x(x - 2)^2(x + 2).$$

The fourth chapter, of fifty pages, deals with parallels. Limiting values are discussed, and parallel lines are said to meet at infinity. The propositions are carefully worded, and the various theorems are shown to be all true or all false together. An alternate treatment, based on Playfair's statement of the fifth postulate is also given, but the two procedures should be more sharply separated, especially in the deductions drawn from the sum of the angles of a triangle. The subsequent propositions of the chapter are all discussed by both methods, but the proper selection of the demonstration which belongs to each is pointed out only in the preface.

A particularly instructive feature of the book is the long list of well-chosen examples and original questions, about two thousand being included. Among them are thirty-five on non-euclidean space, which are certainly novel, and of considerable

educational value provided the pupil has not become so saturated with intuitional ideas derived from the first part that he cannot rid himself of them. It seems to me that in the hands of a skillful teacher the book would prove of value, although the only test is that of actual use. A school text in geometry which succeeds in making a direct and natural transition from the traditional programme to such books as von Staudt's and as Enriques's will be a welcome addition to our mathematical literature.

VIRGIL SNYDER.

Zwölf Vorlesungen über die Natur des Lichtes. Von J. CLASSEN. Leipzig, G. J. Göschen, 1905. x + 249 pp.

The author has already published through Göschen in the Sammlung Schubert three volumes on electricity, magnetism, and light which appeal to a technical public of mathematicians and physicists. The present volume, which is not in the Sammlung Schubert, is of a wholly different character. It is the reprint of a series of lectures given before a lay public and consequently freed of all technicalities whether mathematical or physical. The aim of the book is to set forth and discuss those simple experiments which begin with showing the rectilinear propagation of light and advance systematically and logically to the end of demonstrating the electromagnetic nature of light. A large range of phenomena is treated, including even the latest researches, such as those of Rubens, which put the electromagnetic theory upon a seemingly sure footing. A mathematical treatment of the subject built up in the same straightforward and logical manner would be highly useful, and better worth our attention here.

E. B. WILSON.

NOTES.

THE ninth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY will be held at Stanford University on Saturday, February 24. Abstracts of papers intended for presentation should be in the hands of the Secretary of the Section, Professor G. A. Miller, Stanford University, as early as February 10.

AT the recent meeting of the Chicago Section, Professor T. F. HOLGATE, who has served as Secretary since the organization of the Section in 1897, withdrew from that office, which was thereupon filled by the election of Professor H. E. SLAUGHT. The next meeting of the Section falls on Saturday, April 14.

THE opening (January) number of volume 7 of the *Transactions of the American Mathematical Society* contains the following papers: "On the relation between the three parameter groups of a cubic space curve and a quadric surface," by A. B. COBLE; "On certain hyperabelian functions which are expressible by theta series," by J. I. HUTCHINSON; "On the form of a plane quintic curve with five cusps," by P. FIELD; "The symbolic treatment of differential geometry," by A. W. SMITH; "Groups whose orders are powers of a prime," by W. B. FITE; "Differential parameters of the first order," by H. MASCHKE; "The Kronecker-Gaussian curvature of hyperspace," by H. MASCHKE; "Groups containing only three operators which are squares," by G. A. MILLER; "Theorems converse to Riemann's on differential equations," by D. R. CURTISS; "General mean value and remainder theorems," by G. D. BIRKHOFF; "Determination of the abstract groups of order p^2qr; p, q, r being distinct primes," by O. E. GLENN; "Note on the differential invariants of a surface and of space," by C. N. HASKINS; "On improper multiple integrals," by JAMES PIERPONT.

THE opening (January) number of volume 28 of the *American Journal of Mathematics* contains: "On the quaternary linear homogeneous groups modulo p of order a multiple of p," by L. E. DICKSON; "On the integration of a system of differential equations in kinematics," by J. EIESLAND; "On the determination of the properties of the nodal curve of a unicursal ruled surface," by C. H. SISAM; "Certain surfaces with plane or spherical lines of curvature," by L. P. EISENHART; "The motion of a solid in an infinite liquid," by A. G. GRANVILLE.

THE January number (volume 7, number 2) of the *Annals of Mathematics* contains: "The harmonic analysis of the semicircle and of the ellipse," by A. E. KENNELLY; "Note on the possible number of operators of order 2 in a group of order 2^m," by G. A. MILLER; "A geometrical problem connected with the continuation of a power series," by H. MASCHKE; "On

the determination of a catenary with given directrix and passing through two given points," by H. F. MACNEISH; "Concerning the discontinuous solution in the problem of the minimum surface of revolution," by H. F. MACNEISH; "Introduction to the theory of Fourier's series" (to be continued), by M. BÔCHER.

AT the meeting of the London mathematical society held on December 14 the following papers were read: "On well-ordered aggregates," by Professor C. A. DIXON; "Tables of coefficients for Lagrange's interpolation formula," by Col. R. L. HIPPISLEY: "On the representation of certain asymptotic series as convergent continued fractions," by Professor L. J. ROGERS; "On a new cubic connected with the triangle," by Mr. H. L. TRACHTENBURG; "Some difficulties in the theory of transfinite numbers and order types," by Hon. B. A. W. RUSSELL; "The imaginary in geometry," by Mr. J. L. S. HATTON.

THE third annual meeting of the association of Ohio teachers of mathematics and science was held at the Ohio State University, Columbus, December 28, 1905. Two sessions were held in each section. The following papers in mathematics were read: "Do the mathematical courses in literary colleges properly fit for the mathematics of engineering?" by C. HORNUNG; "Recent contributions marking a real advance in the theory of mathematics teaching, by Miss A. H. PALMIÉ; "Sir Isaac Newton, an estimate," by C. L. ARNOLD; "A contribution from non-euclidean geometry to school spherics," by G. B. HALSTED; "Symbolism in mathematics," by Miss H. E. GLAZIER; "The character of mathematics teaching in the high school," by W. T. HEILMAN; "The influence of college entrance certificates on the teaching of mathematics in the high schools," by H. E. GILES; "Report of committee on syllabus," by G. B. HALSTED. The officers for the present year are: president, W. H. WILSON; vice-president, J. O. WILLIAMS; secretary, T. E. MCKINNEY.

AT the last meeting of the central association of science and mathematics teachers, the following officers were elected: president, O. W. CALDWELL; secretary, C. M. TURTON; treasurer, C. W. D. PARSONS. The officers of the mathematics section are: chairman, J. V. COLLINS; vice-chairman, H. E. SLAUGHT; secretary, Miss MABEL SYKES.

THE mathematical section of the California teachers association held two meetings at Berkeley on December 26 and 28. The following papers were read: "How to cultivate the power to think mathematically," by I. STRINGHAM; "Vital questions for teachers of secondary mathematics," by J. B. CLARKE. During the business meeting the section adopted *School Science and Mathematics* as its official journal, and re-elected the following officers: president, G. A. MILLER; vice president, W. H. BAKER; secretary, J. F. SMITH.

AT the meeting of the mathematics section of the Indiana state teachers association held at Indianapolis December 27, a committee of five with Professor D. A. ROTHROCK, of Indiana University, as chairman was appointed to perfect a plan of organization of the mathematics teachers of the state into an Indiana association of mathematics teachers. The proposed association will meet for organization at Indianapolis, March 30.

THE *Annals of Mathematics*, in pursuance of the policy announced in 1899, when the responsibility for the journal was assumed by Harvard University, has published a considerable number of articles whose purpose is not so much to present new results as to render more accessible, or put in better form, some important portion of mathematical theory. It may not be generally known that reprints of most of these expository articles can be obtained from the Publication Office of Harvard University, 2 University Hall, Cambridge, Mass., at prices ranging from thirty-five to seventy-five cents. The list of titles is as follows:

Galois's theory of algebraic equations. By JAMES PIERPONT. 67 pages.

The theory of linear dependence. By MAXIME BÔCHER. 16 pages.

Sufficient conditions in the calculus of variations. By W. F. OSGOOD. 25 pages.

Space of constant curvature. By F. S. WOODS. 42 pages.

The integral as the limit of a sum, and a theorem of Duhamel's. By W. F. OSGOOD. 18 pages.

The continuum as a type of order: an exposition of the modern theory. With an appendix on the transfinite numbers. By E. V. HUNTINGTON. 63 pages.

Introduction to the theory of Fourier's series. By MAXIME BÔCHER. (In press.)

The following pamphlets are also on sale at the same place:

Regular points of linear differential equations of the second order. By MAXIME BÔCHER. 23 pages.

Introduction to infinite series. By W. F. OSGOOD. 71 pages.

THE following books are announced by the firm of B. G. Teubner, Leipzig, to be in press, and will probably appear in a few weeks: Encyklopädie der mathematischen Wissenschaften, II, 1; III, 1, 2; IV, 2; V, 1; VI, 1, 2; Encyclopédie des sciences mathématiques, tome I, volumes 1, 2, 3, 4; H. POINCARÉ, Der Wert der Wissenschaft, translated by E. WEBER; Serret-Harnack, Lehrbuch der Differential- und Integralrechnung, in 3 Bänden, dritte Auflage, by G. SCHEFFERS. The German translation of Enriques's Questioni riguardanti la geometria elementare, by H. FLEISCHER, will be enlarged to two volumes, of which the second, Die geometrischen Aufgaben, is in the press. The first volume is promised for next fall.

AT the annual public meeting of the Paris academy of sciences, held on December 18, 1905, the following prizes were awarded for mathematical memoirs: the Francoeur prize (fr. 1000) to H. STOUFF; the Poncelet prize (fr. 2000) to N. LALLEMAND; two Petit d'Ormay prizes (each fr. 10000), one to É. BOREL, the other to M. COSSANTIN. The subjects for which these prizes were offered were announced in the BULLETIN, volume 11, page 327.

UNIVERSITY OF PARIS. (The following courses are given during the present semester.)—By Professor G. DARBOUX: General principles of differential geometry, curvilinear coördinates and quadratic differential forms, two hours.—By Professor E. GOURSAT: Elements of the theory of analytic functions, two hours.—By Professor P. PAINLEVÉ: General laws of motion and of equilibrium, two hours.—By Professor P. APPELL: Elements of mathematics preparatory to mechanics and physics, two hours.—By Professor L. RAFFY: Geometric applications of analysis, two hours.—By Professor H. POINCARÉ: Planetary perturbations and development of the perturbation function, two hours.—By Professor J. BOUSSINESQ: Mechanical properties of fluids, two hours.—By Professor E. BOREL: Selected chapters in the theory of integral

functions, one hour. — By Professor G. KOENIGS: General kinematics, two hours. — By Dr. P. PUISEUX: Stars and eclipses, one hour. Mathematical conferences will be held by Professors Goursat, Hadamard, Raffy, Drs. Blutel and Servant.

OXFORD UNIVERSITY (*Hilary term*). — By Professor W. ESSEN: Comparison of analytic and synthetic methods in the geometry of conics, two hours; Synthetic geometry of cubics, one hour. — By Professor E. B. ELLIOTT: Elements of elliptic functions, two hours; Theory of numbers, II, one hour. — By Professor H. H. TURNER: Elementary mathematical astronomy, two hours. — By Professor A. E. H. LOVE: Theory of potential, two hours; Elements of the calculus, two hours. — By Mr. J. W. RUSSELL: Algebra of quantics, two hours. — By Mr. P. J. KIRKBY: Higher algebra, one hour. — By Mr. A. L. DIXON: Calculus of finite differences, one hour. — By Mr. J. E. CAMPBELL: Differential geometry, two hours. — By Mr. C. H. SAMPSON: Solid geomety, II, two hours. — By Mr. A. L. PEDDER: Trigonometry, one hour. — By Mr. C. LEUDESDORF: Geometry, two hours. — By Mr. A. E. JOLLIFFE: Analytic geometry, II, two hours. — By Mr. R. F. MCNEILE: Integral calculus, two hours.

UNIVERSITY OF INNSBRUCK. (The following courses are announced for the summer semester, 1906.) — By Professor K. ZINDLER: Analytic geometry of two and three dimensions, II, four hours; Selected chapters of line geometry, three hours. — By Professor J. MENGER: Descriptive geometry with exercises, II, four hours. The successor of the late Professor O. Stolz has not yet been appointed.

DR. H. HAHN, of the technical school at Vienna, has charge of the courses which were announced for the late Professor Otto Stolz for the present semester at the University of Innsbruck.

DR. F. COHN has been promoted to an associate professorship of mathematics and astronomy at the University of Königsberg.

DR. H. VON KOCH, of the University of Stockholm, has been appointed professor of pure mathematics at the technical school of Stockholm.

PROFESSOR R. MÜLLER, of the technical school of Berlin, has been granted a leave of absence for the present academic year.

DR. G. FABER has been appointed docent in mathematics at the technical school at Karlsruhe.

DR. E. STÜBLER has been appointed docent in mathematics at the technical school of Berlin.

PROFESSOR A. PRINGSHEIM, of the University of Munich, and Professor R. FRICKE, of the technical school at Brunswick, have been elected corresponding members of the Göttingen academy of sciences.

PROFESSOR K. BOEHM, of the University of Heidelberg, and Professor J. HURWITZ, of the technical school at Zürich, have been granted leave of absence for the present semester on account of ill health.

PROFESSOR G. BAUER, of the University of Munich, celebrated his eighty-fifth birthday November 18.

MR. RICHARD MORRIS, instructor in mathematics at Rutgers College, has been promoted to an associate professorship of graphics and applied mathematics.

PROFESSOR C. J. JOLY, of the University of Dublin, and royal astronomer of Ireland, died in December, 1905, at the age of 42 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ACHSEL (R.). Ueber den Zahlbegriff bei Leibniz. Wilmersdorf, 1905. 4to. 20 pp. M. 1.50

ARNDT (K.). Grundbegriffe der höheren Mathematik für Chemiker. Berlin, Mayer, 1905. 8vo. 60 pp. M. 1.50

BORTOLOTTI (E.). Lezioni sul calcolo degli infinitesimi date nella R. Università di Modena. Modena, 1905. 8vo. 7 + 61 pp. L. 3.00

ERTELT (G.). Synthetische Beweise einiger Sätze aus der Theorie der Flächen zweiten Grades. (Progr.) Bielitz, 1905. 8vo. 21 pp.

FULLER (J.). New and original treatise on the geometrical development of round and oval cones. New York, 1905. 8vo. 53 pp. Cloth. $1.25

GREUL (A.). Ueber Scharen von ∞^{2n} Kurven im R_{n+1}. Erster Teil: Der Fall $n=2$. (Diss., Greifswald.) Dresden, Teubner, 1905. 8vo. 32 pp.

HEPKE (B.). Ueber kürzeste Transversalen zwischen Erzeugenden einer hyperboloidischen Regelschaar. (Diss.) Breslau, Fleischmann, 1905. 8vo. 78 pp. M. 2.00

HORN (J.). Gewöhnliche Differentialgleichungen beliebiger Ordnung. Leipzig, Göschen 1905. 12mo. Cloth. (Sammlung Schubert L.) M. 10.00

KOPPISCH (A.). Zur Invariantentheorie der gewöhnlichen Differentialgleichung zweiter Ordnung. (Diss.) Greifswald, 1905. 8vo. 34 pp.

LAMPE (E.). Rede zur Gedächtnisfeier für Guido Hauck am 17. Mai 1905 in der Halle der Technischen Hochschule zu Charlottenburg gehalten. Nebst der Rede am Sarge in der Halle des Friedhofs der Zwölfapostelgemeinde gehalten am 28. Januar 1905 von A. Parisius. Mit einem Bild von G. Hauck als Titelbild. Leipzig, Teubner, 1905. M. 0.80

LOBATCHEWSKY (N. I.). Pangéométrie ou précis de géométrie fondée sur une théorie générale et rigoureuse des parallèles. Nouvelle édition conforme à la première (1835). Paris, Hermann, 1905. 8vo. Fr. 5.00

LODGE (A.). Integral calculus for beginners. London, Bell, 1905. 16mo. 203 pp. Cloth. 4s. 6d.

MILLER (J. R.). Significance of the mathematical elements in the philosophy of Plato. Chicago, 1905. 8vo. 96 pp. $1.00

MÖLLERS (B.). Ueber Normalsysteme, die mit der Rotations- und Schraubenfläche der Traktrix zusammenhängen. Münster, 1905. 8vo. 85 pp. M. 2.50

MÜLLER (E.). See SCHRÖDER (E.).

PARISIUS (A.). See LAMPE (E.).

PETRI (K.). Ueber die in der Theorie der ternären kubischen Formen auftretenden Konnexe. (Diss.). München, 1903. 8vo. 44 pp.

PIRONDINI (G.). Osservazioni relative all'integrazione delle equazioni differenziali delle varie specie. Memoria. Parma, Rossi-Ubaldi, 1905. 8vo. 44 pp.

REMOUNDOS (G.). Sur les zéros d'une classe de fonctions transcendantes. (Thèse.) Paris, Gauthier-Villars, 1905. 4to. 80 pp.

SCHRÖDER (E.). Vorlesungen über die Algebra der Logik (exakte Logik). (In 3 Bänden.) Vol. II: Zweite (Schluss-) Lieferung. Herausgegeben im Auftrage der Deutschen Mathematiker-Vereinigung von E. Müller. Mit einem Bildnisse des Verfassers. Leipzig, Teubner, 1905. 8vo. 32 + 206 pp.

SCHUBERT (H.). Die Ganzzahligkeit in der algebraischen Geometrie. Den Teilnehmern der 48. Versammlung Deutscher Philologen und Schulmänner zu Hamburg 1905 als Festgabe dargeboten von dem Ortskomitee. Leipzig, Spamer, 1905.

SCHUH (F.). Vergelijkend overzicht der methoden ter bepaling van aantallen vlakke krommen. Amsterdam, 1905. 8vo. 11 + 218 pp. M. 6.00

STAUDE (O.). Analytische Geometrie des Punktes, der geraden Linie und der Ebene. Ein Handbuch zum Gebrauche neben den akademischen Vorlesungen. Leipzig, Teubner, 1905. 8vo. 8 + 448 pp. Cloth. (B. G. Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band XVI.) M. 14.00

SÜSS (A.). Die Gruppen, die mit der allgemeinen projektiven Gruppe der Ebene gleiche Zusammensetzung haben. (Diss., Greifswald.) Dresden, Teubner, 1905. 8vo. 34 pp.

TANNERY (J.). Leçons d'algèbre et d'analyse, à l'usage des élèves de la classe de mathématiques spéciales. (En 2 volumes.) Vol. I. Paris, Gauthier-Villars, 1906. 8vo. 7 + 423 pp. Fr. 12.00

WIELEITNER (H.). Theorie der ebenen algebraischen Kurven höherer Ordnung. Leipzig, Göschen, 1905. 12mo. Cloth. (Sammlung Schubert XLIII.) M. 10.00

WOLFF (G.). Ueber Gruppen der Reste eines beliebigen Moduls im algebraischen Zahlkörper. (Diss.). Göttingen, 1905. 8vo. 46 pp. M. 1.50

II. ELEMENTARY MATHEMATICS.

BAUR (L.). Methodisch geordnete Aufgaben aus dem Gebiete der Planimetrie und Stereometrie mit Angabe und Begründung der Berechnungsformeln für Schul- und Selbstunterricht. Stuttgart, Steinkopf, 1905. 8vo. 6 + 167 pp. M. 2.40

BORCHARDT (G. W.). Key to a new trigonometry for schools. Part 2. London, Bell, 1905. 8vo. 244 pp. 5s.

BORTOLOTTI (E.). Aritmetica generale ed algebra per la 1ª classe liceale. Roma, Segati, 1905. 8vo. 120 pp. L. 1.50

BOURLET (C.). Précis d'algèbre, contenant 573 exercices et problèmes, rédigé conformément aux programmes du 31 mai 1902 et du 27 juillet 1905. Paris, Hachette, 1905. 16mo. 445 pp. Fr. 2.50

CABREIRA (A.). Note sur les rapports des solides. Coimbra, 1905. 8vo. 11 pp.

CONANT (L. L.). Original exercises in plane and solid geometry. New York, American Book Company, [1905]. 12mo. 124 pp. Cloth. $0.50

DASSEN (C. C.). Tratado elemental de algebra de acuerdo con el concepto moderno de esta ciencia y los métodos mas rigurosos. Buenos Ayres, Hermanos, 1905. 12mo. 18 + 528 pp.

——. Tratado elemental de geometría euclídea de acuerdo con las ideas modernas y métodos mas rigurosos. Buenos Ayres, Hermanos, 1905. 12mo. 15 + 470 pp.

DAVISON (C.). Elements of solid geometry. Cambridge University Press, 1905. 8vo. 130 pp. 2s. 6d.

FECHNER (H.). Aufgaben für den Unterricht in der Buchstabenrechnung (Algebra). 5te vermehrte Auflage. Berlin, Schultze, 1905. 8vo. 4 + 266 pp. M. 2.50

GRÉVY (A.). Algèbre, à l'usage des élèves des classes de quatrième B à première C et D. 4e édition. Paris, Vuibert et Nony, 1905. 16mo. 379 pp.

——. Géométrie dans l'espace, conforme au programme du 27 juillet 1905, à l'usage des élèves du deuxième cycle A et B. Paris, Vuibert et Nony, 1905. 16mo. Pp. 281–396.

——. Traité de géométrie, à l'usage des élèves de mathématiques A et B (programme du 31 mai 1902) et des candidats aux Ecoles de Saint-Cyr et navale. Paris, Vuibert et Nony, 1905. 8vo. 8 + 598 pp. Fr. 6.00

GUBLER (E.). Aufgaben aus der allgemeinen Arithmetik und Algebra für Mittelschulen. 3tes Heft. Zürich, 1905. 8vo. 48 pp. M. 0.80

HAWKES (H. E.). Advanced algebra. (Mathematical texts.) Boston, Ginn, 1905. 8vo. 13 + 285 pp. Cloth. $1.50

HECHT (C.). Ueber Notwendigkeit und Gestaltung des Unterrichts in der elementaren Mathematik an der höheren Mädchenschule. Vortrag. Bielefeld, Velhagen, 1905. 8vo. 24 pp. M. 0.50

KAMMER (O.). Inversionen bei Permutationen mit Wiederholung. (Progr.) Alsfeld, 1905. 4to. 21 pp.

KOSMIK (K.). Das Dreieck, vornehmlich betrachtet in Bezug auf die ihm zukommenden besonderen Kreise und deren Radien. (Progr.) Mähr.-Weisskirchen, 1905. 8vo 27 pp.

L. (G.). Repertorio di matematiche e fisica elementari. 2a edizione. Livorno, Giusti, 1906. 16mo. 156 pp. L. 1.00

LIMARZI (N.). Compendio pratico di aritmetica, algebra e geometria ad uso delle scuole tecniche e commerciali. Torino, Bertero, 1905. 8vo. 64 pp.

LODGE (O.). Easy mathematics, chiefly arithmetic; being a collection of hints to teachers, parents, self-taught students and adults, and containing a summary or indication of most things in elementary mathematics useful to be known. London and New York, Macmillan, 1905. 12mo. 15 + 436 pp. Cloth. $1.10

LYMAN (E. A.). Advanced arithmetic. For secondary schools and normal schools. New York, American Book Company [1905]. $0.75

MARTINI ZUCCAGNI (A.). Trattato di algebra complementare ad uso degli istituti tecnici, con molti esercizi risoluti. Livorno, Giusti, 1905. 8vo. 8 + 283 pp. L. 3.00

MATHEMATICAL questions and solutions from "Educational Times." New series. Vol. 8. London, Hodgson, 1905. 8vo. 6s. 6d.

MAYER (J. E.). Das mathematische Pensum des Primaners. Ein Hilfsbuch für den Primaner humanistischer und realistischer Gymnasien, für Techniker, etc., sowie besonders für das Selbststudium. Heft 19, 20: Elemente der Mechanik. Leipzig, Schäfer, 1905. 8vo. 86 pp. M. 2.00

MORAGA Y SÁNCHEZ (F.). Programa de algebra y trigonometría. Madrid, Ratés, 1905. 8vo. 24 pp. F. 1.00

NEWSON (H. B.). Graphic algebra for secondary schools. Boston, Ginn, 1905. 12mo. 19 pp. $0.10

PAHL (F.). See SCHULZE (E.).

PETER (P.). Die Zahlenwelt im Bilde. Eine kleine mathematische Unterhaltung. (Progr.) Feldkirch, 1905. 8vo. 21 pp.

PIERCE (R. M.). Problems of number and measure. Outline and bibliography of the application of the Arabic system of notation to a radix other than ten and account of the power of the resulting new system of non-decimal arithmetic as the basis of a throughgoing reform of the metric systems of the world. New York, 1905. 12mo. 21 pp. Cloth. $1.25

SCHULTZ (E.). Vierstellige Logarithmen der Winkelfunktionen. (a) Von 10″ zu 10″. (b) Von 1′ zu 1′. Essen, Baedeker, 1905. 8vo. 42 pp. M. 0.30

SCHULZE (E.) und PAHL (F.). Mathematische Aufgaben. Ausgabe für Gymnasien. 1ter Teil: Aufgaben aus der Planimetrie und Arithmetik für die Unterstufe. Leipzig, Dürr, 1905. M. 2.40

SMITH (D. E.). A modern American course of study in arithmetic arranged by years. Boston, Ginn, 1905. 16mo. 22 pp.

——. Practical arithmetic. Boston, Ginn, 1905. 12mo. 546 pp. Cloth. $0.65

——. Handbook to Smith's arithmetics. Boston, Ginn, 1905. 12mo. 125 pp.

SOULÉ (G.). Philosophical practical mathematics. New Orleans, 1905. 8vo. 1013 pp. Cloth. $6.00

STEIGER (J.). Der geometrische Unterricht in der Volkschule. Zum Gebrauche in Seminarien und für die Hand des Lehrers methodisch dargestellt. 3te, verbesserte Auflage. Bühl, Konkordia, 1905. 6 + 5 + 69 pp. M. 1.00

STERN (H. A.) and TOPHAM (W. H.). Practical mathematics. London, Bell, 1905. 8vo. 388 pp. Cloth. 4s. 6d.

STRENGER. Ueber halbregelmässige Vielflache. (Progr.) Schwäb. Hall, 1905. 4to. 45 pp.

SUÁREZ SOMONTE (I.). Programa de geometría. Madrid, Ratés, 1905. 8vo. 38 pp. Fr. 1.00

——. Programa de nociones y ejercicios de aritmética y geometría. Madrid, Ratés, 1905. 8vo. 36 pp. (Instituto del Cardenal Cisneros.) Fr. 1.00

TOPHAM (W. H.). See STERN (H. A.).

WORKED EXAMPLES in applied mathematics. (*University Tutorial Series.*) London, Clive, 1905. 8vo. 32 pp. 1s.

YOUNG (G. C. and W. H). The first book of geometry. London, Dent, 1905. 12mo. Cloth.

ZOTTI (P.). I logaritmi nei conteggi commerciali e bancari: teoria e pratica, ad uso delle scuole commerciali e istituti tecnici. Milano e Roma. Albrighi, 1905. 16mo. 64 pp. L. 0.80

ZWERGER (M.). Studien im Gebiete der elementaren Mathematik. (Progr.). Würzburg, 1905. 8vo. 44 pp.

III. APPLIED MATHEMATICS.

ABRAHAM (H.) et LANGEVIN (P.). Les quantités élémentaires d'électricité: ions, électrons, corpuscules. Un volume en deux fascicules. Paris, Gauthier-Villars, 1905. 8vo. 16 + 1138 pp. (Mémoires de la Société de physique, 2e série.) Fr. 35.00

ALGER (P. R.). Hydromechanics: an elementary treatise prepared for the use of the midshipmen. Annapolis, Md., Alger, 1905. 12mo. 145 pp. Cloth. $1.50

BACH (C.). Elastizität und Festigkeit. Die für die Technik wichtigsten Sätze und deren erfahrungsmässige Grundlage. Mit in den Text gedruckten Abbildungen und 20 Tafeln in Lichtdruck. 5te, vermehrte Auflage. Berlin, Springer, 1905. 8vo. 24 + 668 pp. Cloth. M. 18.00

BAKER (W. M.). Elementary dynamics. 2d edition, revised. London, Bell, 1905. 16mo. 318 pp. Cloth. 4s. 6d.

BARTLETT (G. M.). Numerical problems in descriptive geometry for class and drawing-room practice. Ann Arbor, Mich., Bartlett, 1905. 12mo. 62 pp. $0.50

BAUSCHINGER (J.). Die Bahnbestimmung der Himmelskörper. Leipzig, Engelmann, 1906. 8vo. 16 + 653 pp. M. 34.00

BELINFANTE (A. A. C.). De duwkracht. Beschouwingen van stoffen en hun bewegingen als gevolgen van botsende- of duwkracht; verwerping der aantrekkingskracht, zwaartekracht en moleculaire krachten. Westzaan, 1905. 8vo. 186 pp. M. 5.00

BENTABOL Y URETA (H.). Cuestiones astronómicas y de mecánica celeste. Madrid, 1905. 8vo. Fr. 4.00

BLAU (E.). Die Mechanik fester Körper. Lehrbuch in elementarer Darstellung für höhere technische Fachschulen und zum Selbstunterricht, nebst einer Sammlung von 250 aufgelösten Beispielen. Hannover, Jänecke, 1905. 8vo. 7 + 263 pp. M. 6.00

BLYTHE (W. H.). Elementary graphic statics and use of tables of sines and tangents to four places of decimals. London, Simpkin, 1905. 4to. 1s. 6d.

BOUSSINESQ (J.). Calcul du pouvoir refroidissant des courants fluides. Paris, Gauthier-Villars, 1905. 4to. 48 pp. Fr. 2.50

—— Sur l'existence d'un ellipsoide d'absorption dans tout cristal translucide, même sans plan de symétrie ni axe principal, et sur la construction des rayons lumineux dans les milieux opaques. Paris, Gauthier-Villars, 1905. 8vo. 21 pp. Fr. 1.00

—— Propagation des ondes le long d'une colonne liquide compressible, se composant de filets à vitesses inégales et contenue dans un milieu élastique horizontal sans tension longitudinale. Paris, Gauthier-Villars, 1905. 4to. 20 pp. Fr. 1.50

BOVEY (H. J.). Theory of structures and strength of materials. 4th edition, rewritten and enlarged. New York, Wiley, 1905. 8vo. 13 + 968 pp. Cloth. $7.50

CAPELLI (C.). Sulla prospettiva parallela o prospettiva assonometrica; teorema di Pohlke. Milano, Marcolli, 1905. 4to. 18 pp., 1 plate.

CHWOLSON (O. D.). Traité de physique. Ouvrage traduit sur les éditions russe et allemande par E. DAVAUX. Edition revue et considérablement augmentée par l'auteur, suivie de notes sur la physique théorique par E. et F. COSSERAT. Vol. II, 1er fascicule: Emission et absorption de l'énergie rayonnante; vitesse de propagation; réflexion et réfraction. Paris, Hermann, 1905. 8vo. 202 pp. Fr. 6.00

CLERKE (A.). The system of the stars. 2d edition. London and New York, Macmillan, 1905. 8vo. 16 + 403 pp. Cloth. $6.50

COSSERAT (E.), COSSERAT (F.), DAVAUX (E.). See CHWOLSON (O. D.).

ELÉMENTS de mécanique, avec de nombreux exercices. Paris, Poussielgue, 1905. 16mo. 8 + 319 pp.

FOERSTER (W.). Astrometrie, oder die Lehre von der Ortsbestimmung im Himmelsraume, zugleich als Grundlage aller Zeit- und Raummessung. 1tes Heft: Die Sphärik und die Koordinatensysteme, sowie die Bezeichnungen und die sphärischen Koordinatenmessungen. Berlin, Reimer, 1905. 8vo. 180 pp. M. 4.00

FRANK (K. G.). Ueber den Einfluss tiefer Temperaturen auf das Brechungs- und Zerstreuungsvermögen durchsichtiger Körper. (Diss.) München, 1905. 8vo. 52 pp.

FRISCHAUF (J.). Die Gauss-Gibbssche Methode der Bahnbestimmung eines Himmelskörpers aus drei Beobachtungen. Mit einem Anhang zum "Grundriss der theoretischen Astronomie." Leipzig, Engelmann, 1905. 8vo. 5 + 47 pp. M. 1.20

GROSSE (W.). Ionen und Elektronen. Eine kurze Darstellung der Entwickelung und Begründung neuerer Anschauungen, insbesondere der Ionentheorie. Leipzig, Quandt, 1905. 8vo. 5 + 94 pp. M. 2.25

GRÜNEBAUM (E. R. VON). Zur Theorie der Zentrifugalpumpen. Berlin, Springer, 1905. 8vo. 7 + 119 pp. M. 3.00

HARTWIG (T.). Leitfaden der konstruierenden Stereometrie. Darstellung der Raumformen im Schrägbilde, nebst einigen Anwendungen von Schrägbildern auf dem Gebiete der theoretischen und rechnenden Stereometrie, darstellenden Geometrie, Mineralogie, mathematischen Geographie und Physik. Wien, Fromme, 1906. 8vo. 39 pp. M. 1.00

HERZ (N.). Lehrbuch der mathematischen Geographie. Wien, Fromme, 1906. 8vo. 8 + 360 pp. M. 12.00

HESSEN (K.). Die rechnerische Bearbeitung der Messungen von Monddistanzen. (Diss.) Berlin, 1905. 4to. 27 pp.

KEFERSTEIN (H.). Strahlengang und Vergrösserung in optischen Instrumenten. Einführung in die neueren optischen Theorien. Berlin, 1905. 8vo. 42 pp. M. 1.60

KITTL (I.). Die elektromagnetische Wellentelegraphie. Zürich, 1905. 8vo. 155 pp. M. 5.40

LANGEVIN (P.). See ABRAHAM (H.).

LINDERS (O.). Zur Klarstellung der Begriffe Masse, Gewicht, Schwere und Kraft. Leipzig, Jäh, 1905. 8vo. 22 pp. M. 1.00

MADELUNG (E.). Ueber Magnetisierung durch schnellverlaufende Ströme und die Wirkungsweise des Rutherford-Marconischen Magnetdetektors. (Diss.). Göttingen, 1905. 8vo. 38 pp.

MARCHIS (L.). Thermodynamique. Vol II: Introduction à l'étude des machines thermiques. Paris, Gauthier-Villars, 1905. 8vo. 3 + 255 pp. Fr. 5.00

MARSHALL (W. C.). Descriptive geometry, for use of students in engineering at the Sheffield Scientific School of Yale University. New Haven, Yale Coöperative Corporation, 1905. 16mo. 61 pp. Cloth. $1.70

MCLEOD (C. H.). Elementary descriptive geometry. New York, Wiley, 1905. 8vo. 9 + 118 pp. Cloth. $1.50

MERRILL (G. A.). An elementary textbook of theoretical mechanics. New York, American Book Co., 1905. 12mo. 3 + 267 pp. Half-leather. $1.50

MEYER (J.). Die Phasentheorie und ihre Anwendung. Stuttgart, Enke, 1905. 8vo. 50 pp. M. 1.20

MOHR (O.). Abhandlungen aus dem Gebiete der technischen Mechanik. Berlin, Ernst, 1906. 8vo. 9 + 459 pp. M. 15.00

RANDALL (O. E.). Descriptive geometry. A textbook for engineering schools and colleges. Boston, Ginn, 1905. 8vo. 209 pp: Cloth. $2.00

RICHTER (J.). Experimentelle Untersuchung der beim Nachzeichnen von Strecken und Winkeln entstehenden Grössenfehler. (Diss., Giessen.) Naumburg a. S., Lippert, 1904. 8vo. 26 pp.

RUFF (F.). Reference book for statical calculations; rapid statics. Vol. I. London, Spon, 1905. 8vo. 5s.

SILBERNAGEL (E.). Bewegung eines Punktes innerhalb einer nicht homogenen Staubmasse mit cylindrischen Flächen gleicher Dichtigkeit. (Diss.) München, 1905. 4to. 47 pp.

SOECKNICK (K.). Ueber das Saint-Venantsche Problem. Teil II. Königsberg, 1905. 4to. 45 pp. M. 1.40

SPRAGUE (C. E.). The accountancy of investment. Including a treatise on compound interest, annuities, amortisation and the valuation of securities. New York, 1905. 8vo. 83 pp. $3.00

VALLIER (M.). Notes sur la dynamique de l'aéroplane. Paris, 1905. 4to. 98 pp. Fr. 3.50

WIEN (W.). Ueber Elektronen. Vortrag gehalten auf der 77. Versammlung Deutscher Naturforscher und Aerzte in Meran. Leipzig, Teubner, 1905. 8vo. 28 pp. M. 1.00